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# A modified symplectic scheme for seismic wave modeling

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ABSTRACT

Symplectic integrators are well known for their excellent performance in solving partial differential equation of dynamical systems because they are capable of preserving some conservative properties of dynamic equations. However, there are not enough high-order, for example third-order symplectic schemes, which are suitable for seismic wave equations. Here, we propose a strategy to construct a symplectic scheme that is based on a so-called high-order operator modification method. We first employ a conventional two-stage Runge–Kutta–Nyström (RKN) method to solve the ordinary differential equations, which are derived from the spatial discretization of the seismic wave equations. We then add a high-order term to the RKN method. Finally, we obtain a new third-order symplectic scheme with all positive symplectic coefficients, and it is defined based on the order condition, the symplectic condition, the stability condition and the dispersion relation. It is worth noting that the new scheme is independent of the spatial discretization type used, and we simply apply some finite difference operators to approximate the spatial derivatives of the isotropic elastic equations for a straightforward discussion. For the theoretical analysis, we obtain the semi-analytic stability conditions of our scheme with various orders of spatial approximation. The stability and dispersion properties of our scheme are also compared with conventional schemes to illustrate the favorable numerical behaviors of our scheme in terms of precision, stability and dispersion characteristics. Finally, three numerical experiments are employed to further demonstrate the validity of our method. The modified strategy that is proposed in this paper can be used to construct other explicit symplectic schemes.



## 1. Introduction

Seismic wave modeling is a powerful tool that can be used for seismic disaster prediction, earth structure investigations, seismic exploration, and other areas of study related to wave phenomenon. With the developments in computer hardware, solving seismic equations with a direct method has become more and more popular in practical applications. In the past decades, a variety of techniques for spatial discretization have been developed, such as the finite difference method (FDM) (Alford et al., 1974; Virieux, 1984, 1986; Moczo et al., 2000; Etgen and O'Brien, 2007; Liu, 2014; Tan and Huang, 2014a; Wang et al., 2014), the pseudo-spectral method (PSM) (Gazdag, 1981; Kosloff and Baysal, 1982), the finite element method (FEM) (Marfurt, 1984; Ma and Liu, 2006; S. Liu et al., 2014), the spectral element method (SEM) (Patera, 1984; Komatitsch and Vilotte, 1998; Cohen, 2002), the nearly analytic discrete method (NADM) (Yang et al., 2003, 2009, 2010), and the Hamiltonian particle method (HPM) (Takekawa et al., 2012, 2014). Other hybrid or optimized methods are also available (Moczo et al., 1997; Takeuchi and Galler, 2000). Each method has its advantages and disadvantages. Virieux et al. (2011) gave a classical review of these methods in their paper. Here, we focus on time integration.

The most frequently used scheme for time integration is a second-order central difference (CD) (Virieux, 1984, 1986), due to its efficiency and ease of implementation. However, a significant drawback of the CD method is its strong numerical dispersion (H. Liu et al., 2014; Tan and Huang, 2014a). In practice, a significantly finer time increment than what is indicated by the stability limit (the maximum time step for a specific spatial discretization) should be adopted to reduce the temporal error associated with a general spatial discretization. This may lead to a dramatic increase in computing costs. Another favorable temporal scheme is the explicit Newmark scheme (Y. Liu et al., 2014), which is almost as efficient as CD method. When solving a second-order wave equation, the Newmark scheme can simultaneously obtain both the velocity field and the displacement field, and this scheme may be useful for ground motion evaluation (Magnoni et al., 2014). However, this scheme also suffers from numerical dispersions especially for long-time wavefield simulations (S. Liu et al., 2014).

An obvious strategy to reduce temporal errors is to use high-order temporal schemes. However, high-order schemes generally correspond to higher computational costs. To balance the competing goals of computational precision and efficiency, Dablain (1986) first applied the so-called Lax–Wendroff method (LWM) to obtain fourth-order

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**Table 1**
Stability parameters for 1D–3D elastic wave equations. M1 is applied for time integration, and spatial derivatives are approximated by central difference operators.

| Dimension and stability parameters | | The order of central difference | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | ∞ |
| 1D | b | 1.7321 | 1.5 | 1.4090 | 1.3586 | 1.3258 | 1.3025 | 1.2850 | 1.2711 | 1.1027 |
| 2D | a | 1 | 0.7586 | 0.5530 | 0.4431 | 0.3743 | 0.3286 | 0.2926 | 0.2668 | 0 |
| | b | 1.7321 | 1.4656 | 1.3367 | 1.2598 | 1.2077 | 1.1697 | 1.1402 | 1.1166 | 0.7800 |
| 3D | a | 1.3385 | 0.7428 | 0.5378 | 0.4315 | 0.3421 | 0.3122 | 0.2879 | 0.2550 | 0 |
| | b | 1.6420 | 1.3014 | 0.1161 | 1.0816 | 1.0281 | 0.9910 | 0.9630 | 0.9396 | 0.6366 |

temporal accuracy. The essential idea of LWM is to replace the high-order temporal derivatives with high-order spatial derivatives based on the Taylor expansion. In this case, only two matrix–vector multiplications (the spatial discretization operator can be written as a matrix form in the general case) can achieve fourth-order accuracy. Another advantage of the fourth-order LWM over the CD is that the stability limit of the former method is $\sqrt{3}$ times the stability limit of the latter method. The increased stability limit allows for a larger time step, which partially compensates for the increased computational costs. In the past few years, many types of LWM have been developed, and they utilize various spatial techniques to evaluate high-order spatial derivatives, such as PSM (Chen, 2009; Long et al., 2013), FEM (De Basabe and Sen, 2010), SEM (De Basabe and Sen, 2010), and NADM (Tong et al., 2013). Recently, Liu and Sen (2013) proposed an arbitrary even-order accuracy scheme in both the time and space domains for 2D acoustic wave equations, which is based on a dispersion-relation-based method and uses an elaborate design of the finite difference stencil. The extension to the 3D acoustic wave equations and the staggered grid stencil is straightforward (Tan and Huang, 2014a,b). Tan and Huang (2014a) proved that this dispersion-relation-based method is the same as the LWM. Although the dispersion-relation-based method and its optimized type have achieved excellent performance for modeling acoustic wave propagation, their extension to elastic equations may be extremely difficult because their dispersion relations may not be explicitly expressed in a general case.

Compared to explicit methods, implicit methods can adopt a relatively larger time step, which has attracted some researchers who would like to apply implicit methods to seismic wave modeling (Luo et al., 2000; Yang et al., 2009, 2010). Implicit methods, such as the second-order implicit symplectic method based on Pade's approximation, the implicit Runge–Kutta scheme, and the Adam scheme, involve the inversion of Laplacian matrix to solve the acoustic wave equation. Although direct LU decomposition yields an exact decomposition, it requires an unaffordable amount of computational costs when it is applied to large-scale seismic wave modeling. Although the spectral or the hybrid methods can reduce computational costs, it is different to satisfy the requirements of high-precision seismic modeling and migration by these methods due to their low accuracy (Luo et al., 2000).

Chen (2007) mainly discussed explicit high-order temporal integrators for solving the acoustic wave equation, such as LWM, Nyström, and the splitting methods. A detailed investigation indicated that the symplectic Nyström and the splitting methods are far superior to the LWM in terms of suppressing numerical errors in long-time computations. This phenomenon is easy to be understood because the symplectic Nyström and splitting methods have respectively fourth- and third-order accuracies for solving the Hamiltonian system of the acoustic wave equation. However, LWM, which implies that the velocity is approximated by a first-order forward difference, has only first-order accuracy for the Hamiltonian system (Lu and Schmid, 1997). Ma et al. (2014) delivered a comparative study of conventional second- to fourth-order symplectic schemes combined with NADM for acoustic and elastic wave modeling. Theoretical analyses and numerical experiments showed that second-order partitioned Runge–Kutta (PRK) method was very suitable for wavefield modeling because of its high-efficiency rates. This method only requires two times matrix–vector multiplications for each time step. Li et al. (2012) developed a so-called symplectic discrete singular convolution differentiator (SDSCD) to simulate elastic wave propagation, and it employs a convolution differentiator for spatial derivatives and a three-stage third-order PRK for the temporal derivative. The SDSCD is very powerful because it can not only capture weak wave responses, but also is appropriate for long-time simulations. Nissen-Meyer et al. (2008) implemented a fourth-order symplectic scheme for elastodynamic spectral element method, which requires four times matrix–vector multiplications in each time step. The computational costs of the fourth-order symplectic scheme may be extremely large compared with a second-order symplectic scheme. The computational costs of the three-stage third-order symplectic scheme may be moderate because this method provides a compromise between numerical accuracy and computational costs. However, in the literature, explicit symplectic schemes, which require three times matrix–vector multiplications, are not common (Ruth, 1983; Chen, 2007; Li et al., 2012). In this study, we focus our attention on developing a new third-order symplectic scheme to simulate seismic wave propagation.

Our strategy for exploring symplectic schemes is to add additional terms into the conventional symplectic schemes. Specifically, we selected a two-stage RKN to illustrate our strategy, and then we added a square operator in terms of spatial discretization to the second equation of RKN. Six symplectic coefficients should obey the order and symplectic conditions, and we found one degree of freedom. We defined the coefficients according to the stability condition and the dispersion relation, and we obtained two sets of solutions, which respectively corresponded to the schemes that had the largest stability range (the domain from 0 to stability limits) and the lowest amount of numerical dispersion, respectively. We suggest one solution for practical use. The newly obtained symplectic scheme is independent of the spatial discretization operator that is used. In this case, we simply chose FDM to allow for straightforward discussions and comparisons. The properties of our scheme are compared with the properties of conventional symplectic and non-symplectic schemes in terms of the stability range and the amount of numerical dispersion. The large stability limits and the low amount of dispersion associated with this scheme suggest that it will be suitable for practical use. Finally, numerical experiments were conducted to verify our theoretical analysis.

## 2. Derivation of the modified symplectic scheme

The medium of the real earth is extremely complex, and it includes viscosity, porosity, and anisotropy (Ma and Liu, 2006; Magnoni et al.,

**Table 2**
Stability parameters for 2D elastic wave equation. M1–M5 schemes are applied for time integration, and a tenth-order central difference operator is used for spatial approximation.

| Stability parameters | M1 | M2 | M3 | M4 | M5 |
|---|---|---|---|---|---|
| a | 0.3743 | 0.3753 | 0.3753 | 0.3754 | 0.3754 |
| b | 1.2077 | 0.6973 | 0.9294 | 0.9018 | 0.6039 |