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Elastic properties of rocks containing oriented systems of ellipsoidal inclusions

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ABSTRACT

In the present work the effective elastic moduli of a rock containing one or more systems of parallel inclusions of 3D-ellipsoidal shape were calculated. The calculations were performed for the conditions of constant strain and stress at infinity. The results were obtained for non-interacting inclusions (low concentration). The presence of movable fluid in the rock is described by the universal Gassmann relations for an anisotropic medium. The comparison of the results obtained with the experimental data has shown that it is possible to apply the proposed calculation technique for the determination of elastic moduli and acoustic waves' velocities in rocks, for example, for double-porosity carbonate formations.

1. Introduction

The determination of elastic parameters of rocks by their microstructure data is one of the most important problems of petrophysics and physics of rocks. There exists a variety of different calculation methods from empirical models to direct modeling by numerical methods of the processes of elastic wave propagation in a rock of given microstructure. Nowadays the optimal methods for the solution of the inverse problem of the interpretation of geophysical measurement data are the effective medium methods. On the one hand, these methods allow considering the geometry of inclusions that the empirical methods don't; on the other hand, they don't require as much computational resources as the methods based on direct numerical solution of the equations of continuum medium mechanics.

A review of the micromechanical methods for the calculation of elastic properties of composite materials is given in the following monographs (Kanaun and Levin, 2008; Nemat–Nasser and Hori, 1998), and the applications of these methods in the physics of rocks are given, for example, in Zimmerman (1991), Berryman (1995), Grechka and Kachanov (2006).

An approach based on micromechanical methods was applied to the modeling of the physical properties of carbonate rocks with double porosity (Kazatchenko et al., 2006a) and the solution of the inverse problems of petrophysics (Kazatchenko et al., 2006b).

As a rule, during the modeling of elastic properties of cracked rocks by micromechanical methods it is assumed that the cracks are penny-shaped, while the concentration of cracks in a unit volume is characterized by a dimensionless parameter — crack density. In fact the most popular models in geophysics are the Hudson (1990) and Thomsen (1995) models and non-interacting inclusion model (Sayers and Kachanov, 1995), which are applicable for crack-like inclusions with "soft" filling. Generally, the calculations of these models are performed for parallel penny-shaped inclusions that lead to effective transversely isotropic medium. An interesting inclusion-based anisotropic poroelasticity model is presented by Xu (1998). Unlike many other authors Xu has not used the dry rock frame moduli as input parameters, but he has calculated them from the pore parameters and properties of the matrix and fluid. The calculations have been performed for spheroidal inclusions (Xu, 1998) and the conditions of constant load and displacement.

Meanwhile, existing experimental data show that, for example, cracked granites are characterized by hexagonal anisotropy. Many inclusions in rocks have finite volume and complex shape (Fig. 1), so the calculations of the physical properties of rocks are realized more adequately in the terms of volumetric concentration than in the terms of crack density. Exactly, the volumetric concentration of inclusions is of major interest during the data treatment of petrophysical measurements in the solution of the problems of oil and gas physics and hydrogeology. As a rule, in natural rocks the volumetric concentration of vugs and cracks (the secondary pores) isn't high that allows us to use relatively simple micromechanical methods developed for low inclusion concentrations.

In the present work the calculations of the elastic properties of rocks containing non-interacting inclusions represented by 3D-elipsiods are given. This model is more general than the conventional model of spheroidal inclusions.

The calculations were performed for the systems of inclusions filled with fluids or elastic materials. The comparison of the results with

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Fig. 1. Example of a double-porosity carbonate rock.

existing experimental data for rocks containing systems of oriented cracks or similar inclusions was performed.

2. Model description

Let us consider an elastic medium (matrix) that is described by the elastic moduli tensor $\boldsymbol{C}$ or the compliance tensor $\boldsymbol{D}$, so that the relation between the stress σ_{ij} and strain $\varepsilon_{ij} = (u_{i,j} + u_{j,i}) / 2$ tensors, where u_i are the displacement vector components, takes the following form:

$$\sigma_{\mathrm{ij}} = C_{\mathrm{ijkl}}\varepsilon_{\mathrm{kl}}, \tag{1}$$

$$\varepsilon_{\mathrm{ij}} = D_{\mathrm{ijkl}}\sigma_{\mathrm{kl}}, \tag{2}$$

where the summation over the repeated subscript k and l is implied.

Ellipsoidal inclusions described by the elastic moduli tensor $\boldsymbol{C^\alpha}$ or the compliance tensor $\boldsymbol{D^\alpha}$ are embedded in this matrix. In order to obtain expressions for the effective compliance of the medium with inclusions let us use the exact geometric relations (Kanaun and Levin, 2008; Nemat-Nasser and Hori, 1998):

$$(\boldsymbol{D}-\overline{\boldsymbol{D}}) : \boldsymbol{\sigma}_0 = \sum_{\alpha=1}^{N} f_\alpha (\boldsymbol{D}-\boldsymbol{D^\alpha}) : \overline{\boldsymbol{\sigma^\alpha}}, \tag{3a}$$

or in the component form

$$\left(D_{\mathrm{ijkl}}-\overline{D_{\mathrm{ijkl}}}\right)\sigma_{0\mathrm{kl}} = \sum_{\alpha=1}^{N} f_\alpha \left(D_{\mathrm{ijkl}}-D^{\boldsymbol{\alpha}}_{\mathrm{ijkl}}\right)\overline{\sigma_{\mathrm{kl}}{}^{\boldsymbol{\alpha}}}, \tag{3b}$$

where $\overline{\boldsymbol{\sigma}^\alpha} = <\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}^d>_\alpha$, $\boldsymbol{\sigma}_0$ is the given uniform stress field; $\boldsymbol{\sigma}^d$ are the perturbations introduced to this stress field by inclusions; $\overline{\boldsymbol{D}}$ is the effective compliance tensor to be found, f_α is the concentration of the α^{th} inclusion. If the inclusion concentration is low enough, the average stress for each inclusion is approximated as a uniform stress of an isolated inclusion placed in a matrix with given stress tensor $\boldsymbol{\sigma}_0$ at infinity.

Let us introduce a tensor $\boldsymbol{A^\alpha} = (\boldsymbol{C} - \boldsymbol{C^\alpha})^{-1} : \boldsymbol{C}$; we can write for an isolated inclusion of α type:

$$\boldsymbol{\varepsilon}^\alpha = <\boldsymbol{\varepsilon}^0 + \boldsymbol{\varepsilon^d}> = \boldsymbol{A^\alpha} : (\boldsymbol{A^\alpha}-\boldsymbol{S^\alpha})^{-1} : \boldsymbol{\varepsilon}_0, \tag{4}$$

where $\boldsymbol{\varepsilon}^\alpha$ is the average strain of each inclusion; $\boldsymbol{S}^\alpha$ is the Eshelby tensor (Eshelby, 1957) of a single inclusion.

The explicit expressions for the components of the Eshelby tensor are given in Appendix A.

Expressing the strain tensor $\boldsymbol{\varepsilon}_0$ through the far field stress tensor $\boldsymbol{\sigma}_0$, it is obtained:

$$\overline{\boldsymbol{\sigma^\alpha}} = \boldsymbol{C^\alpha} : \boldsymbol{A^\alpha} : (\boldsymbol{A^\alpha}-\boldsymbol{S^\alpha})^{-1} : \boldsymbol{D} : \boldsymbol{\sigma}_0. \tag{5}$$

Inserting the expression (5) into (3a) and (3b) we obtain:

$$(\boldsymbol{D}-\overline{\boldsymbol{D}}) : \boldsymbol{\sigma}_0 = \left\{\sum_{\alpha=1}^{N} f_\alpha (\boldsymbol{D}-\boldsymbol{D^\alpha}) : \boldsymbol{C^\alpha} : \boldsymbol{A^\alpha} : (\boldsymbol{A^\alpha}-\boldsymbol{S^\alpha})^{-1} : \boldsymbol{D}\right\} : \boldsymbol{\sigma}_0. \tag{6}$$

As the last equality should be fulfilled independently from the magnitude of the given tensor $\boldsymbol{\sigma}_0$, it follows from (6) that the final expression for the calculation of the elastic compliance of the medium with ellipsoidal inclusions is:

$$\overline{\boldsymbol{D}} = \left\{\boldsymbol{I} + \sum_{\alpha=1}^{N} f_\alpha (\boldsymbol{A^\alpha}-\boldsymbol{S^\alpha})^{-1}\right\} : \boldsymbol{D}, \tag{7}$$

where $I_{ijkl} = \frac{\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}}{2}$, δ_{ij} is the Kronecker symbol. It should be noted that in the case of very flat empty inclusions Eq. (7) leads to the Kachanov (1993) and Sayers and Kachanov (1995) results for the so-called non-interacting approximation.

In the case of the macrostrain $\boldsymbol{\varepsilon}_0$ given at infinity, the average strain tensor within an inclusion is approximated by the deformation tensor in a single inclusion within an elastic body with properties of the matrix and by substituting this expression to a deformation field $\boldsymbol{\varepsilon}_0$. In this case the effective elastic moduli tensor $\overline{\boldsymbol{C}}$ is described by the following expression:

$$\overline{\boldsymbol{C}} = \boldsymbol{C} : \left\{\boldsymbol{I} - \sum_{\alpha=1}^{N} f_\alpha (\boldsymbol{A^\alpha}-\boldsymbol{S^\alpha})^{-1}\right\}. \tag{8}$$

In the case of isotropic matrix and inclusions the components of the tensors $\boldsymbol{D}$ and $\boldsymbol{C}$ (we will consider in this paper only this case) are:

$$\begin{aligned} C_{\mathrm{ijkl}} &= \lambda\delta_{\mathrm{ij}}\delta_{\mathrm{kl}} + \mu\left(\delta_{\mathrm{ik}}\delta_{\mathrm{jl}} + \delta_{\mathrm{il}}\delta_{\mathrm{jk}}\right), \\ C^\alpha_{\mathrm{ijkl}} &= \lambda^\alpha\delta_{\mathrm{ij}}\delta_{\mathrm{kl}} + \mu^\alpha\left(\delta_{\mathrm{ik}}\delta_{\mathrm{jl}} + \delta_{\mathrm{il}}\delta_{\mathrm{jk}}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} D_{\mathrm{ijkl}} &= -\frac{1}{2\mu}\frac{\lambda}{(3\lambda + 2\mu)}\delta_{\mathrm{ij}}\delta_{\mathrm{kl}} + \frac{1}{4\mu}\left(\delta_{\mathrm{ik}}\delta_{\mathrm{jl}} + \delta_{\mathrm{il}}\delta_{\mathrm{jk}}\right), \\ D^\alpha_{\mathrm{ijkl}} &= -\frac{1}{2\mu^\alpha}\frac{\lambda^\alpha}{(3\lambda^\alpha + \mu^\alpha)}\delta_{\mathrm{ij}}\delta_{\mathrm{kl}} + \frac{1}{4\mu^\alpha}\left(\delta_{\mathrm{ik}}\delta_{\mathrm{jl}} + \delta_{\mathrm{il}}\delta_{\mathrm{jk}}\right), \end{aligned} \tag{10}$$

where λ,μ and $\lambda^\alpha,\mu^\alpha$ are the Lame constants of the matrix or inclusions, respectively.

The most difficult part of the calculations by the formulas (7) and (8) is in operation of a 4th rank tensor inversion. In order to execute this operation we have used the transformation of a