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Szegő transformations and Nth order associated polynomials on the unit circle

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ABSTRACT

In this paper we analyze the Stieltjes functions defined by the Szegő inverse transformation of a nontrivial probability measure supported on the unit circle such that the corresponding sequence of orthogonal polynomials is defined by either backward or forward shifts in their Verblunsky parameters. Such polynomials are called anti-associated (respectively associated) orthogonal polynomials. Thus, rational spectral transformations appear in a natural way.

1. Introduction

Spectral transformations have been studied in the literature in the framework of bispectral problems and the factorization of Jacobi matrices [1]. In particular, three canonical perturbations of nontrivial probability measures supported on the real line yield linear spectral transformations:

(i) Christoffel transform: $d\tilde{\mu} = (x-\beta)d\mu, \beta \notin \text{supp}(\mu)$.
(ii) Uvarov transform: $d\tilde{\mu} = d\mu + M\delta(x-\beta), M \in \mathbb{R}_+, \beta \in \mathbb{R}$.
(iii) Geronimus transform: $d\tilde{\mu} = \frac{1}{x-\beta}d\mu + M\delta(x-\beta), M \in \mathbb{R}_+, \beta \notin \text{supp}(\mu)$.

Pure rational transformations appear when a backward (resp. forward) shift is considered in the Jacobi matrix associated with a nontrivial probability measure supported on the real line (see [2,3]). Furthermore, every rational spectral transformation can be obtained as a superposition of linear and associated elementary transformations [3].

More recently, some analog problems have been considered for perturbations of nontrivial probability measures supported on the unit circle (see [4–9]) where linear spectral transforms for the corresponding Carathéodory functions appear in a natural way.

Pure rational spectral transformations for Carathéodory functions have been studied in [10] when backward (resp. forward) perturbations in the Verblunsky parameters associated with a nontrivial probability measure supported on the unit circle are introduced.

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The aim of this contribution is the analysis of pure rational spectral transformations associated with nontrivial probability measures supported on the interval $[-1, 1]$, which are defined by the Szegő antitransformation of the measures considered in [10].

The structure of the manuscript is as follows. In Section 2 we give a basic background about polynomials orthogonal with respect to nontrivial probability measures supported on the real line and the unit circle, respectively. The Szegő transformation between nontrivial probability measures supported on $[-1, 1]$ and the unit circle is introduced. In Section 3, we consider a forward shift in the sequence of Verblunsky parameters associated with a nontrivial probability measure supported on the unit circle. Using the Szegő inverse transformation, we obtain the parameters of the three-term recurrence relation for the corresponding sequence of orthogonal polynomials. The connection between the Stieltjes functions for the initial and perturbed measures is stated. Thus, a pure rational spectral transformation appears. In Section 4, we deal with an analog analysis when a backward shift in the sequence of Verbunsky parameters is introduced.

2. Background and preliminary results

2.1. Orthogonal polynomials on the real line and Stieltjes functions

Let μ be a positive, nontrivial Borel measure, supported on a subset E of the real line. The sequence of polynomials $\{p_n\}_{n\geqslant 0}$, with

$$p_n(x) = \gamma_n x^n + \delta_n x^{n-1} + \cdots, \quad \gamma_n > 0, \tag{1}$$

is said to be an orthonormal polynomial sequence associated with μ if

$$\int_E p_n(x)p_m(x)\mathrm{d}\mu(x) = \delta_{m,n}, \quad m, n \geqslant 0.$$

The corresponding monic orthogonal polynomials, i.e. with leading coefficient equal to 1, are defined by

$$P_n(x) = \frac{p_n(x)}{\gamma_n}.$$

The sequence $\{p_n\}_{n\geqslant 0}$ satisfies the following three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0, \tag{2}$$

with $a_n = \frac{\gamma_{n-1}}{\gamma_n} > 0, n \geqslant 1$, and $b_n = \frac{\delta_n}{\gamma_n} - \frac{\delta_{n+1}}{\gamma_{n+1}}, n \geqslant 0$. The matrix representation of (2) is

$$xp(x) = Jp(x),$$

where $p(x) = [p_0(x), p_1(x), \ldots]^t$ and J is a tridiagonal symmetric matrix

$$J = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \ddots \\ 0 & 0 & a_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

which is called Jacobi matrix, in the literature [11]. There exists a similar expression to (2) using the monic orthogonal polynomials. In such a case, the matrix representation is

$$xP(x) = \tilde{J}P(x),$$

where $P(x) = [P_0(x), P_1(x), \ldots]^t$ and $\tilde{J}$ is a tridiagonal matrix

$$\tilde{J} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ a_1^2 & b_1 & 1 & 0 & \cdots \\ 0 & a_2^2 & b_2 & 1 & \ddots \\ 0 & 0 & a_3^2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

which is called a monic Jacobi matrix.