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Fourier–Hilbert versus Hartley–Hilbert transforms with some geophysical applications

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ABSTRACT

An intimate mathematical relation between Hartley and Hilbert transforms is given here in contrast with the well known Fourier and Hilbert transform relations. It is interesting to note that the Fourier–Hilbert and Hartley–Hilbert transforms while possessing the same magnitude differ in phase by 270°. The inverse Hartley–Hilbert transform returns the original function unlike the Fourier–Hilbert transform which results the negative of the original function. Further, it may be realized that the envelope defined here of the analytic signal in both Fourier–Hilbert and Hartley–Hilbert domains numerically remain the same while differing in polarity. The feasibility of Hartley–Hilbert transform for a straight forward interpretation, total magnetic anomaly due to a thin plate from Tejpur, India and self potential data of the Sulleymonkey anomaly in the Ergani Copper district, Turkey are illustrated in contrast with the Fourier–Hilbert transform. This pair of transforms have innumerable geophysical applications.

1. Introduction

The Hilbert transformer or wide band 90 phase shifter also called a quadrature filter is a device in the form of a linear two-port whose output signal is a Hilbert transform of the input signal. Generally, a signal given by the Hilbert transform of the another signal arises in many systems and devices such as in speech and image processing, modulators, radar systems, measurement systems, network theory, sampling devices, horizontal and vertical component of magnetic field on the surface of the earth, orthogonal components of seismic wave field and many other systems. The input and output signals of the Hilbert transform are frequently a pair of quadrature signal.

The theory of Hilbert transform is closely tied to the theory of Fourier transforms, the first and best known of all integral transforms. However, the Fourier transform being a complex tool, the computation and implementation of Hilbert transform via the frequency domain is not as economical as using the Hartley transform which is a real replacement and an attractive alternative to the well known Fourier transform (Bracewell, 1983 and Sundararajan, 1995). Precisely for this reason, the Fourier and Hartley transforms are being coined as a mathematical twin (Sundararajan, 1997).

Thus, the Hilbert transform assumes a new dimension when it is defined via the Hartley transform instead of the traditional Fourier transform. The complex analytic signal turns out to be real when it is defined in Hartley domain (Soo-Chang and Sy-Been-Jaw., 1990). Such an analytic signal is operated entirely in the real domain, though it does not correspond to the usual concept of the complex analytic signal (Millane, 1994). However, there exists an interesting feature that the amplitude/magnitude of these two versions of the Hilbert transforms are the same while the phase differs by 270° which is explained mathematically hereunder. It may be inferred that the Hartley–Hilbert transform is similar to the Sundararajan transform (Sundararajan et al., 2000).

It may be mentioned here that the Fourier-Hilbert transform has been extensively used in the interpretation of geophysical data over the last three decades particularly in gravity, magnetic and self potential data interpretation (Sundararajan, 1983; Srinivas, 1998 and Sundararajan et al., 1998).

2. Fourier–Hilbert and Hartley–Hilbert transforms:

It is well known that the Hilbert transform of a real function f(t) can be defined as (Thomas, 1969):

$$fh(t) \;=\; \frac{1}{\pi}\int_{0}^{\infty}[FI(\omega)\cos(\omega t) - FR(\omega)\sin(\omega t)]\,d\omega \tag{1}$$

where

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\,. \\ &= FR(\omega)–iFI(\omega). \end{aligned} \tag{2}$$

FR(ω) and FI(ω) being the real and imaginary components of the Fourier transform of f(t).

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On the other hand, the Hilbert transform of f(t) via the Hartley transform can be defined as:

$$hh(t) = -\frac{1}{\pi}\int_0^{\infty}[HO(\omega)\cos(\omega t) + HE(\omega)\sin(\omega t)]d\omega \quad (3)$$

where

$$H(\omega) = \int_{-\infty}^{\infty} f(t)\,\mathrm{cas}(\omega t)dt = HE(\omega) + HO(\omega) \quad (4)$$

HE(ω) and HO(ω) being the even and odd components of the Hartley transform of f(t) while cas(ωt) = cos(ωt) + sin(ωt) is a 45° phase shifted sine wave. The Fourier and Hartley transforms are related through:

$$FR(\omega) = HE(\omega) \quad \text{and} \quad -FI(\omega) = HO(\omega).$$

The Hartley–Hilbert transform hh(t) is similar to the Sundararajan transform defined by Sundararajan et al (2000). The negative sign associated with the Eq. (3) is optional. Further, the two versions of the Hilbert transforms yield the same amplitude while differs in phase by 270°.

3. Analytical signal

The well known Fourier analytic signal and its amplitude of the real function f(t) is expressed as:

$$fa(t) = f(t) - i\,fh(t)$$
$$AF(t) = \sqrt{f(t)^2 + fh^2(t)} \quad (5)$$

where fh(t) is the Fourier–Hilbert transform of f(t).

The Fourier spectra of the complex analytic signal defined by Eq. (5) is given as:

$$FA(\omega)\begin{cases} = 2F(\omega) & \omega > 0 \\ = F(\omega) & \omega = 0 \\ = 0 & \omega < 0 \end{cases}. \quad (6)$$

The spectrum of the Fourier analytic signal implies that we double the +ve frequencies, cancel the −ve frequencies and leave the d.c. level unchanged.

Further, if F(ω) is the Fourier transform of f(t), then the Fourier transform of fh(t) is given by (Thomas, 1969):

$$FH(\omega) = i\,\mathrm{sgn}(\omega)F(\omega)$$

where,

$$\mathrm{sgn}(\omega)\begin{cases} = 1 & \text{for } \omega > 0 \\ = 0 & \text{for } \omega = 0 \\ = -1 & \text{for } \omega < 0 \end{cases} \quad (7)$$

and $i = \sqrt{-1}$.

This is a simple way of determining the Fourier transform of fh(t) in contrast with the direct approach.

On the other hand, the Hartley analytic signal in the Hartley domain is given as:

$$ha(t) = f(t) - i\,hh(t) \quad (8)$$

where hh(t) is the Hartley–Hilbert transform of f(t).

The spectrum of the real Hartley analytical signal defined by Eq. (8) is expressed as:

$$HA(\omega)\begin{cases} = H(\omega) + H(-\omega) & \text{for } \omega > 0 \\ \quad 2HE(\omega) & \\ = H(\omega) & \text{for } \omega = 0 \\ = H(\omega) - H(-\omega) & \text{for } \omega < 0 \\ -2\,HO(\omega) & \end{cases} \quad (9)$$

OR

$$\begin{matrix} 2HE(\omega) \\ H(\omega) \\ 2HO(\omega) \end{matrix}\begin{cases} = 2FR(\omega) & \text{for } \omega > 0 \\ = F(\omega) & \text{for } \omega = 0 \\ = -2FI(\omega) & \text{for } \omega < 0 \end{cases}. \quad (10)$$

If H(ω) and f(t) form the Hartley transform pair, then the Hartley transform of hh(t) is given as:

$$HH(\omega) = \mathrm{sgn}(\omega).H(-\omega)$$

where

$$\mathrm{sgn}(\omega)\begin{cases} = 1 & \text{for } \omega > 0 \\ = 0 & \text{for } \omega = 0 \\ = -1 & \text{for } \omega < 0 \end{cases}. \quad (11)$$

The spectrum of the Hartley analytic signal differs from that of the Fourier analytic signal in that we double +ve frequencies of the even component, leave the dc level unchanged and again double the −ve frequencies of the odd component. The utility of the real Hartley analytic signal [Eq. (8)] is yet to be made explicit in the literature.

The amplitude of the analytical signal of Hartley–Hilbert transform can be given as:

$$AH(t) = \sqrt{f^2(t) + hh^2(t)}. \quad (12)$$

The phase, instantaneous frequency and the envelope are expressed as:

$$PH(t) = \arctan\left(\frac{-hh(t)}{f(t)}\right) \quad (13)$$

$$IFH(t) = \frac{d}{dt}\{PH(t)\} \quad (14)$$

where PH(t) is the phase, IFH(t) is the instantaneous frequency which is the time derivative of the phase and EH(t) is the envelope of the Hartley–Hilbert transform.

$$EH(t) = \frac{f(t)}{\mathrm{Cos}[PH(t)]}. \quad (15)$$

Thus, the amplitude, phase, instantaneous frequency and envelope are auxiliary functions associated with the Hartley–Hilbert transforms similar to the relations as found in Fourier- Hilbert transforms (Sundararajan, 1983). Further, it can be realized that the amplitude AH(t) is numerically is the same as that of envelope EH(t), however differs in sign.

4. Hartley–Hilbert transform relations

If the function f(t) is causal, i.e. f(t) = 0 for t<0, then the even HE(ω) and odd HO(ω) components of the Hartley transform of f(t)