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Inversion of ray velocity and polarization for elasticity tensor

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ABSTRACT

We construct a method for finding the elasticity parameters of an anisotropic homogeneous medium using only ray velocities and corresponding polarizations. We use a linear relation between the ray velocities and wavefront slownesses, which depends on the corresponding polarizations. Notably, this linear relation circumvents the need to use explicitly the intrinsic relation between the wavefront slowness and ray velocity, which – in general – is not solvable for the slownesses. We discuss sensitivity of this method to the errors in measurements.

1. Introduction

In this paper, we discuss a method of using ray-velocity and polarization measurements to determine the complete set of the density-scaled elasticity parameters that describe a Hookean solid. The problem of determining the twenty-one components of the elasticity tensor from wave propagation has been investigated by many researchers; among them, van Buskirk et al. (1986), Norris (1989), and Dewangan and Grechka (2003). In all cases, the proposed methods relied on using polarizations and wavefront slownesses. We propose a method for finding these components using polarizations and ray velocities. To find the latter quantities, we consider traveltimes measured between a single point source and point receiver, which are directly related to the ray velocities. This method circumvents the need to measure the wavefront slownesses, which – in a seismological context – requires closely spaced sources or receivers.

We also demonstrate that standard seismic measurements of polarizations and traveltimes allow us to obtain uniquely the density-scaled elasticity parameters. In view of the forward problem described in the next section and the inversion formulated in Section 3, we can infer that the relation of the elasticity parameters to the polarization and traveltime measurements is one-to-one in the context of the theory of elastodynamics.

In Section 4, we discuss the error analysis for the proposed method and exemplify it with a numerical example.

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2. Waves and rays in Hookean solids

A Hookean solid is fully described by its mass density and elasticity parameters, which are components of the elasticity tensor appearing in a constitutive equation. The constitutive equation of a Hookean solid is[1]

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}, \quad i,j,k,l \in \{1,2,3\},$$

where σ, c and ε are stress, elasticity and strain tensors, respectively. Strain tensor is a symmetric second-rank tensor given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

where u and x are the displacement and position vectors, respectively. Elasticity tensor is a fourth-rank tensor that possesses the intrinsic symmetries, $c_{ijkl}=c_{jikl}=c_{klij}$, and is positive-definite, $c_{ijkl}v_iw_jv_kw_l>0$, for nonzero vectors v and w. We will consider homogeneous media, where c and mass density, ρ, do not depend on position. In such a case, the signal propagation is described by the following form of the elastodynamic equations:

$$\frac{c_{ijkl}}{\rho}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = \frac{\partial^2 u_i}{\partial t^2}.$$

These equations depend on the density-scaled elasticity tensor, $a := c/\rho$, whose components are the parameters that fully describe a medium in the context of this paper. The asymptotic solutions of these equations lead to the Christoffel equations,

$$\Gamma_{ik}(p)A_k(p) = A_i(p) \tag{1}$$

[1] We use repeated indices summation convention throughout this paper.

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where $\Gamma_{ik}(p) := a_{ijkl}p_jp_l$,$p$ is the slowness vector normal to the wavefront and A is the polarization vector. Christoffel matrix $\Gamma(p)$ is positive-definite due to the positive definiteness of the elasticity tensor. The nontrivial solution of the Christoffel equations requires that

$$\det(\Gamma(p) - I) = 0,$$

which can be written as

$$\left(H^1(p) - 1\right)\left(H^2(p) - 1\right)\left(H^3(p) - 1\right) = 0$$

where $H^\alpha(p)$,$\alpha \in \{1,2,3\}$, are the three eigenvalues of $\Gamma(p)$. Each of the terms set to zero is the eikonal equation for one of the three waves. For each eikonal equation, its characteristic equations are

$$\begin{aligned} \dot{x}_i^\alpha &:= \frac{dx_i^\alpha}{dt} = \frac{1}{2}\frac{\partial H^\alpha}{\partial p_i} \\ \dot{p}_i^\alpha &:= \frac{dp_i^\alpha}{dt} = -\frac{1}{2}\frac{\partial H^\alpha}{\partial x_i}, \end{aligned} \tag{2}$$

which are the Hamilton equations. The solutions of these equations are rays. In homogeneous media, the second equation reduces to $\dot{p}=0$, which implies that the rays are straight lines. Eq. (2) defines the ray velocity, which in the case of straight rays is given by the vector connecting the source and the receiver divided by the traveltime; both the source and receiver locations are known from the experimental setup, and the traveltime is a key measurement.

3. Inversion for density-scaled elasticity parameters

In this section, we derive expressions that allow us to obtain the density-scaled elasticity parameters using the measured traveltimes and corresponding polarizations. The relationship between the ray velocity, $\dot{x}$, and wavefront slowness, p, is given by Eq. (2). Using this equation, it is, in general, impossible to express p as an explicit function of $\dot{x}$ and a. In our formulation, we circumvent this problem by including the measured polarization, A, and expressing p in terms of $\dot{x}$, a and A.

To do so, we consider the Christoffel Eq. (1),

$$a_{ijkl}p_jp_lA_k(p) = A_i(p),$$

which describes polarization A as an eigenvector of the Christoffel matrix, Γ. The Christoffel equation restricts the eigenvalues to the slowness surfaces given by

$$H^\alpha(p) = 1.$$

Since $H^\alpha(p)$ is an eigenvalue of $\Gamma(p)$ with corresponding unitary eigenvector $A^\alpha(p)$, we write

$$\Gamma_{ik}(p)A_k^\alpha(p) = H^\alpha(p)A_i^\alpha(p); \tag{3}$$

multiplying by $A_i^\alpha(p)$, we write

$$a_{ijkl}p_jp_lA_i^\alpha(p)A_k^\alpha(p) = H^\alpha(p), \tag{4}$$

since $|A^\alpha(p)|=1$. Using expression (4) in Eq. (2), we write

$$\dot{x}^{i\alpha} = a_{jikl}p_lA_j^\alpha(p)A_k^\alpha(p) + a_{mjkl}p_jp_l\frac{\partial A_m^\alpha(p)}{\partial p_i}A_k^\alpha(p).$$

Using Eq. (3), we get

$$\dot{x}^{i\alpha} = a_{jikl}p_lA_j^\alpha(p)A_k^\alpha(p). \tag{5}$$

This equation can be found, among others, in the classic book of Červený (2001, pp. 150–151). Eq. (5) can be written as

$$\dot{x}^\alpha = \Gamma(A^\alpha(p))p.$$

Since $\dot{x}$ and A are obtained from the measurements, Eq. (5) form a linear system for p. The solution exists since the determinant of this system, $\det(a_{jikl}A_j^\alpha A_k^\alpha)$, is nonzero due to the positive definiteness of the Christoffel matrix. The solution is

$$p^\alpha = \Gamma^{-1}(A^\alpha)\dot{x}^\alpha. \tag{6}$$

This equation expresses the wavefront-slowness vector as a function that is linear in the ray velocity and quadratic in the polarization. The dependence on the density-scaled elasticity tensor is contained in Γ. For a wavefront given by a fixed α, we obtain the measurements of $\dot{x}$ and A that result in p. In view of multiple arrivals of the same wave, from now on, the index α distinguishes among different wavefront arrivals for a single source–receiver pair, and not among the three waves, as was the case above. Still, to obtain all twenty-one parameters of elasticity, we need to consider the measurements corresponding to the three waves since, for a given wave, some of the components of a might not appear in Eq. (8), as exemplified by transverse isotropy: the expressions for the traveltime and the polarization of the fastest wave do not contain a_{1212} even though the transversely isotropic medium is described by a_{1111},a_{1133}, a_{3333},a_{2323} and a_{1212}, see, e.g., Slawinski (2003, pp. 229–232).

Combining expression (6) with the Christoffel Eq. (1), we obtain a as an implicit function of $\dot{x}$ and A, which we write as

$$\Gamma\left(\Gamma^{-1}(A^\alpha)\cdot x^\alpha\right)A^\alpha = A^\alpha. \tag{7}$$

Writing the inverse of Γ using its minors, we can rewrite expression (7) as

$$a_{ijkl}\mathrm{Minor}(\Gamma(A^\alpha))_{jm}\dot{x}_m^\alpha\mathrm{Minor}(\Gamma(A^\alpha))_{\ln}\dot{x}_n^\alpha A_k^\alpha = (\det\Gamma(A^\alpha))^2A_i^\alpha, \tag{8}$$

where $i \in \{1,2,3\}$. For a given α and a fixed source–receiver pair, this is an implicit system of three equations for a. These equations are polynomials in components of a, A and $\dot{x}$; they contain only the fifth and the sixth powers of the components of a, the thirteenth powers of components of A, and the second powers of the components of $\dot{x}$ multiplied by the ninth powers of the components of A.

We can view each of the three Eq. (8) as an equation for a hypersurface in the twenty-one dimensional space of density-scaled elasticity parameters. We need to obtain twenty-one such hypersurfaces. From each measurement, we obtain three hypersurfaces. If we measure all three waves for a given source–receiver pair, we obtain at least nine hypersurfaces; more than nine if there are multiple arrivals of the same wave. In an ideal case with no measurement errors, all hypersurfaces intersect at the point corresponding to the density-scaled elasticity tensor that describes the medium. For the intersection of these hypersurfaces to be zero-dimensional, it is necessary that the normals to these hypersurfaces be linearly independent.

To find the density-scaled elasticity parameters from Eq. (8) we can use regression analysis. In the next section, we discuss the relation between the number of measurements and the accuracy of results. A large number of measurements also ensures that we have enough equations to determine uniquely the density-scaled elasticity tensor.

4. Stability analysis: numerical example

In this section, we discuss the stability of the proposed method; in particular, we discuss the sensitivity of the elasticity parameters to the ray-velocity and polarization measurement errors. To do so, we consider the following density-scaled elasticity tensor.[2]

$$\begin{bmatrix} 4.00 & 2.06 & 2.10 & -0.05 & 0.01 & -0.02 \\ & 3.83 & 1.96 & 0.12 & -0.05 & 0.13 \\ & & 3.96 & 0.11 & 0.03 & -0.09 \\ & & & 1.00 & 0.11 & -0.07 \\ S & Y & M & & 0.88 & 0.01 \\ & & & & & 1.11 \end{bmatrix}\left[\frac{\mathrm{km}^2}{\mathrm{s}^2}\right] \tag{9}$$

[2] These values were used by Dewangan and Grechka (2003).