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An improved formulation of the incompressible Navier–Stokes equations with variable viscosity

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ABSTRACT

We present a new formulation of the incompressible Navier–Stokes equations with variable viscosity. By utilizing the incompressibility constraint to remove the trace from the deviatoric stress tensor, we eliminate second-order cross-derivatives of the velocity field, simplifying and improving the accuracy of co-located discretization techniques on both structured- and unstructured grids. This formulation improves the performance of SIMPLE-type algorithms that use sequential mass-momentum iterations to enforce incompressibility. A trace-free stress tensor also removes a typical source of net-rotation for simulations employing free-slip boundary conditions in spherical geometry. We implement the new scheme as a modification of an existing Boussinesq convection code, which we benchmark against analytical solutions of the Stokes problem in a spherical shell with both constant and radially dependent viscosity, and time-dependent thermal convection at infinite Prandtl number with large viscosity contrasts.

1. Introduction

Natural convection is a fundamental process across many areas of science and engineering. The development of accurate, efficient and robust methods for solving the coupled conservation equations of mass, momentum and energy is thus the subject of considerable research.

In many applications, the fluid undergoing convection can be considered incompressible. This leads to a simplification of the governing equations since terms proportional to the divergence of the velocity field vanish. For highly viscous fluids, an additional simplification applies when the Prandtl number (the ratio of viscous to thermal diffusion rates) can be taken as effectively infinite. Under this approximation, known as Stokes or creeping flow, the fluid has negligible inertia, which allows one to neglect the non-linear term describing the self-advection of the velocity field. Finally, if the characteristic density variations in the system under study are small enough to be negligible except when they act as source of buoyancy, we arrive at the incompressible, infinite Prandtl number, Boussinesq equations for natural convection (provided in Section 2). A useful application of these equations is the modeling of the dynamics and long-term thermal evolution of the rocky mantles of terrestrial planets and the icy shells of outer solar system bodies. Numerical simulations based on the solution of such equations are today one of the primary tools for studying these processes.

The dynamics of creeping convection in planetary applications depends strongly on the temperature- and stress-dependent rheology of mantle rocks (e.g. Karato, 2010). The resulting sharp viscosity gradients represent a difficult challenge for numerical models and thus require special attention. During the last 30 years, a variety of numerical codes have been developed that focus on the issue of dealing with large viscosity contrasts (e.g. Baumgardner, 1985; Tackley, 1996; Choblet et al., 2007; Burstedde et al., 2008; Tackley, 2008; Zhong et al., 2008).

In this work we describe a reformulation of the Navier–Stokes equations that we implement in a finite-volume code that refines the previous work of Hüttig and Stemmer (2008a). By explicitly incorporating the incompressibility constraint into the momentum equation, we show that cross-derivatives of the velocity field can be eliminated in the framework of a co-located discretization scheme, i.e. where all the unknown variables of the problem are solved for at the same grid location. This increases the robustness of the code to large viscosity gradients, saves resources and aids computational speed as all interpolations needed to compute mixed derivatives of the velocity field at cell walls, which can become particularly demanding in the presence of unstructured grids (Hüttig and Stemmer, 2008b,a), become no longer necessary. A side

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benefit of our approach is that it naturally eliminates well-known net-rotation problems that routinely arise in spherical or cylindrical convection models with free-slip boundaries (Zhong, 2001; Zhong et al., 2008; Kameyama et al., 2008; Tosi and Yuen, 2011).

The resulting code, named Gaia-v2, is benchmarked against known results. The benchmark tests include Stokes-flow calculations, for which analytical solutions based on the matrix propagator technique are available for constant and radially varying viscosity, as well as complex thermal convection scenarios with temperature dependent viscosity, for which a few numerical results exist that are based on the outcomes of the finite-element code CitComS (Zhong et al., 2008).

2. Equations and model

2.1. Governing equations

The dynamics of planetary mantles and most other natural convection phenomena can be described by conservation of mass, momentum and energy. Here, we use the Boussinesq approximation to account for density variations due to temperature alone, which implies that we deal with an incompressible fluid. Furthermore, we make the infinite-Prandtl-number approximation (Stokes-flow), appropriate for convection in planetary mantles. Under these assumptions, the non-dimensional conservation equations are (e.g. Schubert et al., 2001):

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\nabla \cdot \boldsymbol{\sigma} + Ra\,T\boldsymbol{e}_r = 0, \tag{2}$$

$$\frac{\partial T}{\partial t} = \nabla^2 T - \boldsymbol{u} \cdot \nabla T, \tag{3}$$

where $\boldsymbol{u}$ is the velocity, T temperature, t time, $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}$ the stress tensor, with p dynamic pressure, $\boldsymbol{I}$ identity tensor and $\boldsymbol{\tau}$ deviatoric stress tensor defined as

$$\boldsymbol{\tau} = \mu(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}), \tag{4}$$

where μ is the dynamic viscosity. The body-force term $Ra\,T\boldsymbol{e}_r$ represents a buoyancy force acting in the radial direction $\boldsymbol{e}_r$ due to changes in density solely associated with temperature. The Rayleigh number Ra is a scaling parameter.

2.2. Including mass conservation in the stress tensor

The general constitutive relation for a Newtonian fluid is

$$\boldsymbol{\tau} = \mu\left(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u} + \left(\beta - \frac{2}{3}\right)\nabla \cdot \boldsymbol{u}\boldsymbol{I}\right), \tag{5}$$

where the proportionality factor $(\beta - 2/3)$ accounts for the so-called bulk viscosity β and compressibility. Setting $\lambda \equiv \beta - 2/3$, we can write the stress tensor components explicitly as follows:

$$\boldsymbol{\tau} = \begin{pmatrix} 2\mu\partial_x u + \lambda\nabla\cdot\boldsymbol{u} & \mu(\partial_y u + \partial_x v) & \mu(\partial_z u + \partial_x w) \\ \mu(\partial_x v + \partial_y u) & 2\mu\partial_y v + \lambda\nabla\cdot\boldsymbol{u} & \mu(\partial_z v + \partial_y w) \\ \mu(\partial_x w + \partial_z u) & \mu(\partial_y w + \partial_z v) & 2\mu\partial_z w + \lambda\nabla\cdot\boldsymbol{u} \end{pmatrix}, \tag{6}$$

where u, v and w denote the Cartesian components of the velocity vector $\boldsymbol{u}$. For an incompressible fluid the divergence of the flow field vanishes (Eq. (1)). However, this is only approximately true if one uses a sequential solution method like SIMPLE or one of its modifications to enforce the incompressibility constraint (e.g. Patankar, 1980). These methods decouple pressure and velocity and require an iterative solution of the system until mass and momentum conservation are satisfactorily achieved. The deviatoric stress tensor (6) is expected to be trace-free from the mass conservation constraint (1). Nevertheless, iterative solution schemes leave a non-zero trace from a residual divergence term. While this trace can be significant during the first iterations, it gradually vanishes as convergence is reached.

Due to the incompressibility of the fluid, λ in Eq. (6) can take any value. This is important because choosing $\lambda = -\mu$ not only explicitly removes the trace from the deviatoric stress tensor but also eliminates second-order cross-derivatives in the momentum equation, thereby facilitating the discretization of Eq. (2). The resulting stress tensor reads

$$\boldsymbol{\tau} = \begin{pmatrix} \mu(\partial_x u - \partial_y v - \partial_z w) & \mu(\partial_y u + \partial_x v) & \mu(\partial_z u + \partial_x w) \\ \mu(\partial_x v + \partial_y u) & \mu(\partial_y v - \partial_x u - \partial_z w) & \mu(\partial_z v + \partial_y w) \\ \mu(\partial_x w + \partial_z u) & \mu(\partial_y w + \partial_z v) & \mu(\partial_z w - \partial_x u - \partial_y v) \end{pmatrix}. \tag{7}$$

Taking the divergence of the tensor (7), we obtain, e.g. for the x-component:

$$\begin{aligned}(\nabla\cdot\boldsymbol{\tau})_x = {} & \partial_x\mu(\partial_x u - \partial_y v - \partial_z w) + \mu(\partial_{xx}u - \partial_{yx}v - \partial_{zx}w) \\ & + \partial_y\mu(\partial_y u + \partial_x v) + \mu(\partial_{yy}u + \partial_{xy}v) + \partial_z\mu(\partial_z u \\ & + \partial_x w) + \mu(\partial_{zz}u + \partial_{xz}w).\end{aligned} \tag{8}$$

Collecting μ and defining $\boldsymbol{E} \equiv \nabla(\ln\mu)$, we have:

$$\begin{aligned}(\nabla\cdot\boldsymbol{\tau})_x = {} & \mu\big(E_x(\partial_x u - \partial_y v - \partial_z w) + \partial_{xx}u - \partial_{yx}v - \partial_{zx}w \\ & + E_y(\partial_y u + \partial_x v) + \partial_{yy}u + \partial_{xy}v + E_z(\partial_z u + \partial_x w) \\ & + \partial_{zz}u + \partial_{xz}w),\end{aligned} \tag{9}$$

where the subscripts of the vector $\boldsymbol{E}$ indicate its Cartesian components. Simplifying eliminates second-order cross-derivatives and we obtain:

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau} &= \mu\begin{pmatrix} E_y\partial_x v - E_x\partial_y v + E_z\partial_x w - E_x\partial_z w + \boldsymbol{E}\cdot\nabla u + \nabla^2 u \\ E_x\partial_y u - E_y\partial_x u + E_z\partial_y w - E_y\partial_z w + \boldsymbol{E}\cdot\nabla v + \nabla^2 v \\ E_x\partial_z u - E_z\partial_x u + E_y\partial_z v - E_z\partial_y v + \boldsymbol{E}\cdot\nabla w + \nabla^2 w \end{pmatrix} \\ &= \mu\Big(\boldsymbol{E}\cdot(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u} - \nabla\cdot\boldsymbol{u}\boldsymbol{I}) + \nabla^2\boldsymbol{u}\Big).\end{aligned} \tag{10}$$

Any Laplace operator with variable coefficients can be decomposed into an advection-like part and a pure Laplacian. In Eq. (10), the gradient of the logarithm of viscosity $\boldsymbol{E}$ can be interpreted as a velocity-like field that advects the actual velocity components to counteract the generic smoothing of the Laplacian. If this gradient is large, the system becomes dominated by the advection-like term and numerical methods suitable for generic advection problems (i.e. for hyperbolic partial differential equations) may become more appropriate.

An immediate advantage of the above formulation is the absence of second-order cross-derivative terms. This makes the implementation of the momentum equation with variable viscosity easier in any co-located finite-volume or -difference scheme, particularly on unstructured grids. Clearly, in the case of constant viscosity, the divergence of the stress tensor (10) reduces to the usual vector-Laplacian of the velocity field $\mu\nabla^2\boldsymbol{u}$, so the advantages of our formulation only arise in situations with spatially varying viscosity.

3. Benchmark simulations

For the series of benchmark simulations that will be presented in this section we have solved, in a co-located finite-volume framework, the conservation equations (1) and (2) using the SIMPLE method (Patankar, 1980) by discretizing explicitly the term (10). Additionally, in time-dependent thermal convection runs, we solved Eq. (3) employing a fully-implicit three levels scheme with second-order accuracy in space and time (Harder and Hansen, 2005; Hüttig and Stemmer, 2008a). Both linear systems arising from the discretization of the momentum and energy equations