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Geoid and topography of Earth-like planets: A comparison between compressible and incompressible models for different rheologies

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ABSTRACT

A systematic study of 2D-axisymmetric, spherical shell models of compressible and incompressible mantle convection with constant and variable viscosity and constant and depth-dependent thermodynamic properties is presented. To account for compressibility effects, we employ the anelastic liquid approximation. In the case of variable viscosity, an Arrhenius law with strongly temperature and pressure dependent viscosity is considered. We show that assuming compressible convection with depth-dependent thermodynamic properties strongly influence the geoid undulations. Using compressible convection with constant thermodynamic properties is physically inconsistent and may lead to spurious results for the geoid and convection pattern. In addition, we examine the impact of compressibility as well as different rheologies on the power law relation that connects the Nusselt number to the Rayleigh number. We discover that the power law index of the Nu–Ra relationship is controlled by the rheology, independent of which approximation is used. Instead, the bound of this relation is controlled by a combination of different approximation and rheology.

1. Introduction

Since almost five decades mantle convection has been studied by numerical models. Because of the limitations imposed by finite computing capacity, several simplifying assumptions have commonly been made in most mantle convection studies. One of the most important of such assumptions is the Boussinesq approximation. This approximation is valid if the temperature scale height (i.e. the depth over which temperature increases by a factor of "e" due to adiabatic compression) is much greater than the convection depth (Mihaljan, 1962; Spiegel and Veronis, 1960). However, a temperature scale height in the Earth's mantle is at best only slightly greater than the mantle depth. Hence, the Boussinesq approximation could mask some very important stratification and compressibility effects that influence both the spatial and temporal structure of the convection (Glatzmaier, 1988).

Since the whole mantle thickness of Venus, Earth and Mars are between 45% and 50% of mean planetary radius (Stevenson et al., 1983), global models of mantle convection require a spherical geometry (Bercovici et al., 1992). The pioneer series study of mantle convection in spherical geometry has been done by Zebib (e.g. Zebib et al., 1985; Zebib and Schubert, 1979; Zebib et al., 1983; Zebib et al., 1978, 1980). Following these studies, a number of other studies in spherical geometry focusing on effects such as phase changes, variable viscosity, etc. have been published (see e.g. Schubert et al. (2001) for more references). However, in almost all early models the Boussinesq approximation has been used to simplify the equations governing fluid motion in order to facilitate numerical computation. Different challenges can be encountered if one is interested in the geoid, because the density is the primary variable for the geoid. Therefore, it is important to go beyond the Boussinesq approximation for the geoid computation.

Indeed, the importance of compressible convection has been discovered in two-dimensional Cartesian geometry for iso-viscous convection (Jarvis and McKenzie, 1980) and for variable viscosity (Quareni et al., 1986; Yuen et al., 1987), without looking at the geoid except for a study by Schmeling (1989) who investigated the geoid variation in two-dimensional variable viscosity compressible convection. However, Cartesian geoid is not helpful if to be compared to the real Earth, and spherical models are necessary. Moreover, he computed the geoid undulations by neglecting depth-dependence of the thermodynamic parameters, although the variations of the bulk modulus, thermal expansion and thermal conductivity are known to be large across the Earth's mantle (e.g. Anderson, 1987).

Recently, three-dimensional compressible convection with depth-dependent thermodynamic properties attracted great attention (e.g. Tackley, 1996, 2008). Yet, such studies did not focus on the geoid for compressible cases, especially with depth-dependent properties.

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In the present paper, we will consider compressibility in our mantle convection models, assuming that density varies both radially and laterally, being determined as a function of pressure and temperature through an appropriate equation of the state. Moreover, thermodynamic properties vary with compression and will be considered as a function of depth. Since these depth dependences have a strong effect on the role of top and bottom thermal boundary layers, and the boundaries controls topography and thus the geoid, the effect of compression on the geoid will be significant. The aim of this paper is examining details of the structure of the spherical axi-symmetric Anelastic Liquid Approximation model (ALA) with special attention to the Arrhenius rheology, and compare it to the cases of compressible convection without depth dependent thermodynamical properties, and to cases of the extended Boussinesq approximation. At the same time, the effects of the interaction between temperature and pressure-dependent viscosity and thermodynamic parameters in the compressible mantle convection on the geoid and topography will be studied.

2. Model description

We employ a time dependent, anelastic liquid model (Jarvis and McKenzie, 1980), which lies between Boussinesq and fully compressible representations, to account for the compressibility effects. The anelastic liquid approximation is a very good approximation of the Earth's mantle because typical velocities of the mantle materials are extremely small, so by comparing with seismic wave velocities, we can neglect elastic waves in the mantle convection models.

The equations of mass, momentum and energy of a compositionally homogeneous medium are:

$$\nabla.(\rho u) = 0, \tag{1}$$

$$\rho \frac{Du_i}{Dt} = \rho g_i - \nabla_i p + \frac{\partial}{\partial x_j}\tau_{ij}, \tag{2}$$

$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{Dp}{Dt} = \nabla.k\nabla T + \tau_{ij}\frac{\partial u_i}{\partial x_j} + Q. \tag{3}$$

where t is time, u the velocity, p is the pressure, τ_{ij} is the deviatoric stress, T is the absolute temperature, and c_p, k, α, g, Q are the specific heat capacity, thermal conductivity, thermal expansivity, gravity acceleration, and the rate of internal heating per volume, respectively. These equations are completed by an equation of state that can be written as a linearized Taylor expansion of density about some reference state, ρ_r

$$\rho = \rho_r\left(1 - \alpha(T - T_s) + K_{iso}^{-1}(p - p_h)\right), \tag{4}$$

where T_s is adiabatic temperature distribution and p_h, K_{iso} are the hydrostatic pressure and isothermal incompressibility, respectively.

For a Newtonian fluid in which the bulk viscosity is unimportant, the stress can be related to the velocity field by a constitutive equation such as:

$$\tau_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right), \tag{5}$$

where η is dynamic viscosity and δ_{ij} the Kronecker delta.

The Eqs. (1)–(3) can be non-dimensionalized by introducing the non-dimensional variables (see Table 1) and non-dimensional parameters defined as:

$$\mathrm{Ra}_n = \frac{c_p g \alpha_0 \Delta T \rho_0^2 d^3}{k_0 \eta_{ref}}, \tag{6}$$

$$D_i = \frac{g\alpha d}{c_p}, \tag{7}$$

Table 1
Scaling variables.

Symbol	Quantity	Value
d	Thickness of the mantle	2891 km
ΔT	Temperature difference across the mantle	3380 K
r_0	Radius of the Earth	6371 km
r_{cmb}	Radius of the CMB	3480 km
k_0	Thermal conductivity	$4\,\mathrm{W m^{-1} K^{-1}}$
ρ_0	Density	$4000\,\mathrm{kg\,m^{-3}}$
D_{i0}	Dissipation number	0.56
c_p	Specific heat	$1250\,\mathrm{J\,kg^{-1}\,K^{-1}}$
α_0	Thermal expansion	$2.5\times10^{-5}\,\mathrm{K^{-1}}$
γ_0	Grüneisen parameter	1.4
g	Gravity acceleration	$9.8\,\mathrm{m\,s^{-2}}$

Subscript zero identifies reference value at the top boundary.

where Ra_n is the nominal Rayleigh number which characterizes convection vigor, and D_i is the dissipation number as a measure of the scale height of the adiabatic temperature. d, and ΔT are the convection depth, and temperature difference between top and bottom, respectively. η_{ref} is the reference viscosity corresponding to the reference temperature equal to 1500 K and a reference depth of 100 km. (Please see also appendix A of the author's previous paper, Shahraki and Schmeling, 2012). Indeed, the definition of the nominal Ra number for a system with variable coefficients, in particular, viscosity, is somewhat arbitrary. We choose a definition based on a reference viscosity corresponding to an arbitrary but reasonable sublithospheric reference temperature, and surface values of the other reference state variables. Choosing the surface viscosity as a reference viscosity does not make sense for an Arrhenius-type rheology. An alternative choice of a nominal Ra number would be using reference values of the viscosity and the other state variables at the core mantle boundary (CMB). However, these values are less well known than shallow or surface values. The details for the non-dimensional governing equation can be found in the Schubert et al. (2001).

The specification of the pressure at one point in the system suffices to establish the pressure. On the contrary, to evaluate the velocity field we need additional information. The mechanical boundary conditions at the top and bottom surfaces of the spherical shell are given by zero radial velocities and a shear stress-free condition. No pressure boundary conditions are needed, except that the pressure at one point in the system needs to be fixed. Its value is arbitrary and may be adjusted such as to result in an averaged zero vertical stress at the surface once the topography is calculated from the surface tractions. In addition, we considered the hot inner surface and the cold outer surface as isothermal, and the thermal boundary conditions are given by $T_{bot} = 3653.15\,[\mathrm{K}]$ at the inner surface and $T_{top} = 273.15\,[\mathrm{K}]$ for the outer surface.

2.1. Reference state

The reference state is that of an adiabatic, homogenous fluid under hydrostatic pressure. In this reference state the density distribution, ρ_r, can be obtained as the solution of the Adams-Williamson equation,

$$\frac{1}{\rho_r}\frac{d\rho_r}{dr} = -\frac{\rho_r g}{K_{ad}} = -\frac{\alpha g}{c_p \gamma} \tag{8}$$

where K_{ad} is the adiabatic incompressibility and, γ is the Grüneisen parameter which is a measure of the anharmonic character of the equation of the state (Balachandar et al., 1993) defined as:

$$\gamma = \frac{\alpha K_{ad}}{\rho_r c_p}. \tag{9}$$

According to the condition of $\frac{\alpha}{\gamma} \propto \rho^{-2}$ (Leitch et al., 1992), Zhang and Yuen (1996) simplified this equation and derived: