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Uniqueness of entire functions that share one value ☆

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ABSTRACT

Using Nevanlinna's value distribution theory, we study the uniqueness of entire functions that share only one value and prove some theorems which are related to one famous problem of Hayman [W.K. Hayman, Research Problems in Functions Theory, Athlone Press, London, 1967].

1. Introduction and main results

In this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane C. It is assumed that the reader is familiar with the notations of Nevanlinna's theory such as $T(r,f)$, $N(r,f)$, $m(r,f)$, $\overline{N}(r,f)$ and so on, which can be found, for instance, in [1–4]. We denote by $S(r,f)$ any function satisfying $S(r,f)=o\{T(r,f)\}$, as $r\to+\infty$, possibly outside of a set with finite measure.

Let $f(z)$ be a nonconstant meromorphic function on the complex plane C and $a\in C\bigcup\{\infty\}$. Set $E(a,f)=\{z:f(z)-a=0\}$, where a zero point with multiplicity m is counted m times in the set. If these zero points are only counted once, then we denote the set by $\overline{E}(a,f)$. Let k be a positive integer. Set $E_{k)}(a,f)=\{z:f(z)-a=0, \exists i, 1\le i\le k, \text{ s.t.}, f^{(i)}(z)\neq 0\}$, where a zero point with multiplicity m is counted m times in the set.

Let $f(z)$ and $g(z)$ be two meromorphic functions. We say $f(z)$ and $g(z)$ share the value a CM (counting multiplicities) if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities, i.e., $E(a,f)=E(a,g)$. If we do not consider the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicities), i.e., $\overline{E}(a,f)=\overline{E}(a,g)$.

Moreover, we use the following notations.

Let a be a finite complex number, and k be a positive integer. We denote by $N_{k)}(r,1/(f-a))$ the counting function for the zeros of $f(z)-a$ with multiplicity $\le k$, and by $\overline{N}_{k)}(r,1/(f-a))$ the corresponding one for which the multiplicity is not

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counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for the zeros of $f(z)-a$ with multiplicity $\geq k$, and $\overline{N}_{(k}(r, 1/(f-a))$ be the corresponding one for which the multiplicity is not counted. Set $N_k(r, 1/(f-a)) = \overline{N}(r, 1/(f-a)) + \overline{N}_{(2}(r, 1/(f-a)) + \cdots + \overline{N}_{(k}(r, 1/(f-a))$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $\overline{E}(1, f) = \overline{E}(1, g)$. We denote by $\overline{N}_L(r, 1/(f-1))$ the counting function for 1-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with multiplicity not being counted, and denote by $N_{11}(r, 1/(f-1))$ the counting function for common simple 1-points of both $f(z)$ and $g(z)$. Similarly, we introduce the notation $\overline{N}_L(r, 1/(g-1))$.

Due to Nevanlinna [2], it is well known that if f and g share four distinct values CM, then f is a Möbius transformation of g. Corresponding to one famous question of Hayman [5], Fang and Hua [6], Yang and Hua [7] showed that similar conclusions hold for certain types of differential polynomials when they share only one value. They proved the following result.

Theorem A. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 6$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share* 1 CM, *then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Xu and Qiu [8] improved the above result by deriving the following theorem.

Theorem B. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 12$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share* 1 IM, *then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Recently, Fang [9] proved the following results which were an improvement and generalization of Theorem A.

Theorem C. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and let n, k be two positive integers with $n > 2k+4$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share* 1 CM, *then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Remark 1. Let $k = 1$. Then by Theorem C we get Theorem A.

Theorem D. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k+8$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share* 1 CM, *then $f(z) \equiv g(z)$.*

Now it is natural to ask by Theorems A and B whether the CM sharing value can be replaced by the IM sharing value in Theorems C and D? In this paper, we give a positive answer to the above question by proving the following theorems.

Theorem 1. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and let n, k be two positive integers with $n > 5k+7$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share* 1 IM, *then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Remark 2. Let $k = 1$. Then by Theorem 1 we get Theorem B.

Theorem 2. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n > 5k+13$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share* 1 IM, *then $f(z) \equiv g(z)$.*

Furthermore, based on the idea of multiple values, we obtain the following theorems which improve Theorems C and D respectively.

Theorem 3. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and let n, k, and m be three positive integers. If $E_{m)}\left(1, (f^n)^{(k)}\right) = E_{m)}\left(1, (g^n)^{(k)}\right)$, and*

(i) if $m = 1$ and $n > 4k+6$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$; or

(ii) if $m = 2$ and $n > (5k+9)/2$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$; or

(iii) if $m \geq 3$ and $n > 2k+4$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.

Theorem 4. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and let n, k, and m be three positive integers. If $E_{m)}\left(1, (f^n(f-1))^{(k)}\right) = E_{m)}\left(1, (g^n(g-1))^{(k)}\right)$, and*

(i) if $m = 1$ and $n > 4k+11$, then $f(z) \equiv g(z)$; or
(ii) if $m = 2$ and $n > (5k+16)/2$, then $f(z) \equiv g(z)$; or
(iii) if $m \geq 3$ and $n > 2k+7$, then $f(z) \equiv g(z)$.