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Simulations of fluid motion in ellipsoidal planetary cores driven by longitudinal libration

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ABSTRACT

Deformed by tidal forces, the cavity of a planetary fluid core may be in the shape of a biaxial ellipsoid $x^2/a^2 + y^2/b^2 + z^2/a^2 = 1$, where a and b are two different semi-axes and z is in the direction of rotation. Gravitational interaction between a planet and its parent star exerts an axial torque on the planet and forces its longitudinal libration, a periodic variation of its angular velocity around its rotating axis. Longitudinal libration drives fluid motion in the planetary core via both viscous and topographic coupling between the mantle and fluid. For an arbitrary size of the equatorial eccentricity $\mathscr{E} = \sqrt{a^2 - b^2}/a$, direct numerical simulation of the fully nonlinear problem is carried out using an EBE (Element-By-Element) finite element method. It is shown that fluid motion driven by longitudinal libration vacillates between two different phases: a prograde phase when the planet's rotation speeds up and a retrograde phase when it slows down. For weak longitudinal libration, the fluid motion is laminar without exhibiting noticeable differences between the two phases and a multi-layered, time-independent, nearly geostrophic mean flow can be generated and maintained by longitudinal libration in a biaxial or triaxial ellipsoidal cavity. For strong slow libration, there are profound differences between the two different phases: the retrograde phase is usually marked by fluid motion with instabilities and complex spatial structure while in the prograde phase the flow is still largely laminar.

1. Introduction

As a consequence of the variation of gravitational force across a planet, the form of many planets may be described by a biaxial ellipsoid (for example, Dermott, 1979),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1, \qquad (1)$$

where z is in the direction of rotation, x points to the parent star of the planet, and the semi-axes a and b are different ($a > b$). Gravitational interaction between a planet of biaxial ellipsoidal shape and its parent star can force longitudinal libration, a periodic variation of its angular velocity around its rotating axis (for example, William et al., 2001). Longitudinal libration represents an important dynamic property of many planets which has been employed to verify the existence of liquid layers in planetary interiors and to determine the thickness of the overlying mantle (for example, Margot et al., 2007; Baland and Van Hoolst, 2010). The present study is mainly concerned with the dynamic response of a planetary liquid core, confined in a biaxial or triaxial ellipsoidal cavity, to its longitudinal libration.

The dynamical response of a spherical liquid core ($a = b$) to planetary longitudinal libration is quite different from that of a biaxial ellipsoidal core ($a \neq b$). In a spherical cavity, which is axisymmetric with respect to the rotation axis, the coupling between the librating solid mantle of a planet and its liquid core is purely viscous. A number of authors have investigated librationally driven flows in a spherical cavity. Aldridge and Toomre (1969) (see also Greenspan (1968)) studied axisymmetric inertial oscillations in a librating spherical container experimentally, revealing the resonance of axisymmetric inertial modes at some particular librating frequencies. The problem of spherical libration was also recently investigated by experimental and numerical methods (Noir et al., 2009; Tilgner, 1999; Sauret et al., 2010) and analytically (Busse, 2010). It is the viscous coupling between the solid mantle and the core fluid, via the thin Ekman boundary layer and its mass flux, that plays an essential role in determining the structure and amplitude of the fluid motion driven by longitudinal libration in spherical geometry.

In a biaxial ellipsoidal cavity with $a \neq b$, which is non-axisymmetric with respect to the rotation axis, the fluid core and the solid librating mantle are coupled via both topographical and viscous ef-

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fects. Suppose that the fluid core of viscosity ν is confined within a biaxial ellipsoidal cavity with $b^2 = a^2(1-\mathscr{E}^2)$, where $\mathscr{E}$ is the equatorial eccentricity, and that the planet rotates non-uniformly with mean angular velocity Ω_0. For small equatorial eccentricity $0 < \mathscr{E} \leqslant O(E^{1/4})$, where E is the Ekman number defined as $E = \nu/(a^2\Omega_0)$, the topographic and viscous couplings are equally important in determining the primary properties of librationally driven flows. But typical values of the Ekman number E for many planets are extremely small, $E \leqslant O(10^{-14})$, such that $O(E^{1/4}) \ll \mathscr{E} < 1$. In this case, the topographic coupling between a librating mantle and its liquid core would be predominant although the viscous boundary layer at the core–mantle boundary, together with its nonlinear effect, can play an essential role in generating mean flows in librating biaxial or triaxial ellipsoidal cavities.

In a recent asymptotic analysis of weakly librating flows for a biaxial ellipsoidal cavity with moderate equatorial eccentricity $E^{1/4} \ll \mathscr{E} \ll 1$ (Zhang et al., 2011), it is found that (i) longitudinal libration, via the topographic coupling between the librating mantle and the fluid core, can drive non-axisymmetric oscillatory motion with the azimuthal wavenumber $m = 2$ and (ii) resonance with inertial modes cannot take place at any frequency of libration. The present study will focus primarily on strongly librating flow driven by planetary longitudinal libration in a biaxial ellipsoidal fluid core with moderate or large equatorial eccentricity $E^{1/4} < \mathscr{E} < 1$, in attempting to provide a clear understanding of the topographic and viscous coupling between a librating planet and its liquid core. In the first part of this study, we shall extend the previous asymptotic analysis (Zhang et al., 2011) for a biaxial ellipsoid to that for a triaxial ellipsoidal cavity, which will be validated by fully three-dimensional simulation using an EBE (Element-By-Element) finite element method (Chan et al., 2007). We shall also extend the previous numerical simulation for weak libration, which behaviors almost linearly, to that of the strongly nonlinear regime where secondary or higher instabilities can occur.

In what follows we shall begin by presenting the governing equations in the mantle frame for the longitudinal libration problem in Section 2. A brief discussion of the asymptotic solution for weakly librational flow in a triaxial ellipsoid is presented in Section 3 while the result of direct numerical simulation for both weakly and strongly librational flow is presented in Section 4. A summary and some remarks are given in Section 5.

2. Mathematical formulation of the problem

Consider a viscous, homogeneous fluid of viscosity ν confined in a biaxial or triaxial ellipsoidal cavity. Suppose that the ellipsoidal container rotates rapidly with a non-uniform angular velocity $\mathbf{\Omega}$ given by

$$\mathbf{\Omega} = \hat{\mathbf{z}}[\Omega_0(1+\delta\sin\hat{\omega}t)], \tag{2}$$

where Ω_0 is the mean rate of rotation, $\hat{\mathbf{z}}$ is a unit vector in the direction of rotation, $\hat{\omega}$ is the libration frequency of the planet and $\Omega_0\delta$ represents the dimensional amplitude of longitudinal libration. In a frame of reference attached to the container, the mantle frame, librationally driven flows in the ellipsoidal cavity of an incompressible fluid are governed by the dimensional equations (see, for example, Greenspan, 1968):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + 2(1+\delta\sin\hat{\omega}t)\Omega_0\hat{\mathbf{z}}\times\mathbf{u} + \frac{1}{\rho}\nabla p = \nu\nabla^2\mathbf{u} + \mathbf{r}\times\frac{\mathrm{d}}{\mathrm{d}t}[\Omega_0(1+\delta\sin\hat{\omega}t)\hat{\mathbf{z}}], \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{4}$$

where $\delta > 0$, $\mathbf{r}$ is the position vector, p is a reduced pressure, $\mathbf{u}$ is the three-dimensional velocity field $\mathbf{u} = (u_r, u_\theta, u_\phi)$ with corresponding unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ in spherical polar coordinates (r, θ, ϕ) with $\theta = 0$ at the axis of $\hat{\mathbf{z}}\Omega$ and $r = 0$ at the center of the ellipsoid. The centrifugal force is combined with other conservative forces to form the reduced pressure p. The final term on the right-hand side of (3) is known as the Poincaré force, which drives librational flow against viscous dissipation. The libration amplitude $\Omega_0\delta$ controls the degree of nonlinearity of the problem. It is helpful to distinguish two different phases during a period of longitudinal libration: a prograde phase when the planet rotates faster than the mean rate, marked by $\sin\hat{\omega}t > 0$ in (3), and a retrograde phase when $\sin\hat{\omega}t < 0$. The transition instant between the prograde and retrograde phases, identified by $\sin\hat{\omega}t = 0$, will be referred to as the turning point.

Employing the semi axis a as the length scale, Ω_0^{-1} as the unit of time and $\rho a^2\Omega_0^2$ as the unit of pressure yields the dimensionless equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\hat{\mathbf{z}}\times\mathbf{u} + \nabla p = E\nabla^2\mathbf{u} - 2\delta\hat{\mathbf{z}}\times\mathbf{u}\sin(\hat{\omega}_l t) + (\delta\hat{\omega}_l)(\mathbf{r}\times\hat{\mathbf{z}})\cos(\hat{\omega}_l t), \tag{5}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{6}$$

where the Ekman number, $E = \nu/(\Omega_0 a^2)$, provides the measure of relative importance between the typical viscous force and the Coriolis force, $\hat{\omega}_l = \hat{\omega}/\Omega_0$ is the dimensionless frequency of libration and δ, the Poincaré number or the libration amplitude, quantifies the strength of the Poincaré forcing. Librationally driven flow on the bounding surface, $\mathscr{S}$, of the container is at rest, requiring that

$$\hat{\mathbf{n}}\cdot\mathbf{u} = \hat{\mathbf{n}}\times\mathbf{u} = 0, \tag{7}$$

where $\hat{\mathbf{n}}$ denotes the normal to the bounding surface, $\mathscr{S}$, of the biaxial or triaxial ellipsoidal cavity.

The problem defined by Eqs. (5) and (6) subject to the boundary conditions (7) will be first solved by an asymptotical method for weakly librational flow in a triaxial ellipsoid in Section 3 and by a numerical method via direct nonlinear simulation for both weak and strong longitudinal libration in Section 4.

3. Asymptotic solution in a triaxial ellipsoid

We first extend the previous asymptotic analysis for a biaxial ellipsoid (Zhang et al., 2011) to that for a triaxial ellipsoid, demonstrating that the size of the polar eccentricity is of secondary importance in longitudinally libration-driven flow. Without loss of general physics, we assume that a triaxial ellipsoid is described by the dimensionless equation

$$\frac{x^2}{1} + \frac{y^2}{(1-\mathscr{E}^2)} + \frac{z^2}{(1+\mathscr{E}^2)} = 1, \tag{8}$$

where $E^{1/2} \ll \mathscr{E}^2 \ll 1$, which allows us to ignore the viscous effect to the first approximation. Consequently, the non-slip boundary condition (7) becomes

$$\hat{\mathbf{n}}\cdot\mathbf{u} = 0. \tag{9}$$

For weak longitudinal libration $\delta \ll \mathscr{E}^2$, we seek an asymptotic solution to (5) and (6) by expanding the variables, $\mathbf{u}$ and p, in a series in powers of δ and $\mathscr{E}^2$

$$\mathbf{u} = \delta\mathbf{u}_0 + (\delta\mathscr{E}^2)\mathbf{u}_1 + \cdots, \quad p = \delta p_0 + (\delta\mathscr{E}^2)p_1 + \cdots, \tag{10}$$

where $\mathbf{u}_0$ and p_0 denote the leading-order solution directly driven by the Poincaré forcing while $\mathbf{u}_1$ represents the flow in connection with the topographic effect of a triaxial ellipsoid. We may also expand the normal $\hat{\mathbf{n}}$ of the bounding triaxial ellipsoidal surface $\mathscr{S}$ around the spherical surface $r = 1$, which gives rise to