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The construction of exact Taylor states. II: The influence of an inner core

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ABSTRACT

The geodynamo mechanism, responsible for sustaining Earth's magnetic field, is believed to be strongly influenced by the solid inner core through its influence on the structure of convection within the tangent cylinder. In the rapidly rotating low-viscosity regime of the geodynamo equations relevant to the Earth's core, the magnetic field must satisfy a continuum of conditions known as Taylor's constraint. Magnetic fields that satisfy this constraint, termed Taylor states, have the property that their axial magnetic torque vanishes when averaged over any geostrophic contour, cylinders of fluid coaxial with the rotational axis. In recent theoretical developments, we proved that when adopting a truncated spherical harmonic expansion, the continuous constraint in space reduced to a finite spectral set of conditions. Furthermore, an expedient choice of regular radial basis presents an under-determined problem when constructing Taylor states in a full-sphere showing the ubiquity of such solutions. A spherical-shell geometry, with a conducting inner core, complicates the formulation of Taylor's constraint due to the partitioning of the geostrophic contours into three distinct regions, ostensibly trebling the stringency of the constraint. This raises questions as to the admissible structures of such Taylor states, and their relation to those in a full-sphere. In this paper, we address these issues in two stages. First, we present a mathematical characterisation of the structure of Taylor's constraint in a spherical-shell. We then enumerate the effective number of conditions that must be satisfied by any magnetic field that is everywhere C^∞ inside the core, and show that, assuming an equal truncation in radial and solid angle representation, the number of conditions is approximately 5/3 times that for a full-sphere Taylor state. Second, we investigate the influence of the inner core on the structure of admissible Taylor states by constructing a low-degree family of optimally smooth observationally consistent examples in both a spherical-shell and a full-sphere. We show that the introduction of an inner core into a full-sphere increases the minimum magnetic field complexity, simply by virtue of the increased potency of Taylor's constraint, a trait more pronounced in our quasi-axisymmetric models. We speculate that axisymmetric dynamo-generated exact Taylor states, particularly those generated in a spherical-shell, in general have small radial length scales that may be difficult to resolve.

1. Introduction

The Earth's magnetic field is generated in the fluid outer core by convective flows that twist and stretch magnetic field lines, a process that sustains the field against its natural tendency to decay. Over the last few decades, with the advent of modern computing, huge leaps have been made in understanding the geodynamo, although many questions are left unresolved. One fundamental question is the role played by the solid inner core, particularly in view of its poorly constrained age. The issue is that, on the one hand, the existence of the inner core is believed necessary to produce the thermal and compositional buoyancy necessary to drive convection yet, on the other, thermal history models predict that, of the more than 3 Ga history of the geodynamo, the inner core has only existed for 1 Ga (Buffett, 2003). Aside from the source of convection, there is much interest in the influence of the inner core on the dynamics of the geodynamo, the outer core being split into regions outside and inside the so-called tangent cylinder, an imaginary cylinder parallel to the rotation axis of the Earth and tangent to the inner core. There is a growing body of evidence of strong retrograde vortices inside the tangent cylinder from observations (Hulot et al., 2002), laboratory experiments (Aurnou et al., 2003) and geodynamo models (Sreenivasan and Jones, 2005), suggesting the tangent cylinder marks a separation in the dynamics of the core. Furthermore, the time taken for a magnetic field to diffuse into the electrically conducting inner core is speculated to be enough to stabilise the geodynamo and reduce the frequency of global reversals (Gubbins, 1999; Hollerbach and Jones, 1993).

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Undoubtedly one of the most promising avenues with which to explore the geodynamo is numerical simulations. However, not only are these models difficult to analyse but, using present-day algorithms and computing resources, they are confined to parameters that are far from realistic (Kono and Roberts, 2002). Of particular note is the Ekman number, a nondimensional measure of viscosity, believed to be $O(10^{-15})$ in the core yet models can only reach at most $O(10^{-7})$ (Kageyama et al., 2008; Takahashi et al., 2005, 2008). As well as increasing the role of viscous forces, the associated inflation of the Rossby number, a measure of the strength of inertia, by about 10^3 leads to solutions that operate in a regime distinct from the Earth's core (Christensen and Aubert, 2006; Sreenivasan and Jones, 2006). Despite these shortcomings, geodynamo models have produced, with surprising success, magnetic fields characteristic of the Earth showing features such as realistic field strengths, dipolar dominance and even global reversals. However, few robust conclusions can be drawn due to the strong dependence of model dynamics on the particular choice of the controlling parameters. Even the most basic of properties, the rotationally aligned dipolar structure, defies simple explanation as, depending on the particular choice of parameters, equatorial dipole structures can dominate, particularly for strongly driven convection (Aubert and Wicht, 2004).

The correct dynamics of the Earth's core is described by the magnetostrophic balance between buoyancy, pressure, Coriolis and Lorentz forces (Fearn, 1998). In this regime, where viscosity and inertia do not play a role, J.B. Taylor showed in 1963 that the magnetic field must satisfy a continua of conditions, namely that the axial magnetic torque must average to zero over any geostrophic contour, cylinders of fluid coaxial with the rotation axis. Denoting the magnetic field as $\mathbf{B}$, these constraints assume the form

$$\mathcal{T}(s) \equiv \int_{C(s)} ([\nabla \times \mathbf{B}] \times \mathbf{B})_\phi s \, \mathrm{d}\phi \, \mathrm{d}z = 0 \tag{1}$$

where (s, ϕ, z) are cylindrical coordinates and $C(s)$ denotes a geostrophic contour. In a full-sphere (with no inner core), these contours form a single continuous family of cylinders parameterised by their radius s. However, in a spherical-shell the existence of a solid inner core partitions contours inside the tangent cylinder into two parts: those above and below the inner core. Thus there are three sets of geostrophic contours rather than just one, defined in regions I, II and III of Fig. 1(a); Fig. 1(b) shows example contours for each of the three regions.

Progress in finding any examples of Taylor states, candidates for the structure of the geomagnetic field, has been difficult. The Earth's internal field is believed to be one, although its structure is hidden from view since the only observable is the radial component of the field on the core–mantle boundary (CMB). Current geodynamo models are not yet in an Earth-like parameter regime and cannot produce Taylor states, although there is some evidence that, by measuring the "Taylorisation" of the magnetic field, the correct regime is being approached (Rotvig and Jones, 2002; Takahashi et al., 2005). Progress has been more straightforward when adopting axisymmetry, and several examples of Taylor states have been found by solving the axisymmetric dynamo equations with small Ekman number (Hollerbach and Ierley, 1991; Soward and Jones, 1983). However, since these are all specific cases, it has not been possible to address the broader question of what, if anything, characterises the internal structure of a Taylor state. Furthermore, in view of simplicity these studies did not contain an inner core, leaving open the question of what effect, if any, the change in geometry to a more realistic spherical-shell has on the structure of solutions. Indeed, the fact that Taylor's constraint involves three continuums of conditions rather than just one, leads to the expectation that the class of admissible magnetic field solutions is significantly smaller than that for a full-sphere. The purpose of this paper is to confront this issue head on, by providing an elementary mathematical structure for Taylor's constraint and the explicit construction and comparison of exact spherical-shell with full-sphere Taylor states within a certain well-defined class.

The foundation on which we build was laid down in Livermore et al. (2008), and rests on looking for solutions of (1) in isolation. This more abstract analysis removes the requirement that such magnetic fields are stable, or even time-averaged, solutions of the full set of geodynamo equations in the Earth-like limit. To describe the key result, let us first introduce some notation. We will write the magnetic field in a truncated set of poloidal and toroidal vector spherical harmonics,

$$\mathbf{B} = \sum_{l=1}^{L_{\max}} \sum_{m=0}^{l} \mathbf{S}_l^{m\,s/c} + \mathbf{T}_l^{m\,s/c} \tag{2}$$

where

$$\mathbf{S}_l^{m\,s/c} = \nabla \times \nabla \times [Y_l^{m\,s/c}(\theta, \phi) S_l^{m\,s/c}(r)\hat{\mathbf{r}}],$$
$$\mathbf{T}_l^{m\,s/c} = \nabla \times [Y_l^{m\,s/c}(\theta, \phi) T_l^{m\,s/c}(r)\hat{\mathbf{r}}],$$

in spherical polar coordinates (r, θ, ϕ) and with $\hat{\mathbf{r}}$ denoting the unit position vector. The notation $Y_l^{m\,s/c}$ represents a spherical harmonic of degree l, order m, and azimuthal dependence $\sin m\phi$ or $\cos m\phi$ as appropriate; we adopt the usual Schmidt quasi-normalisation

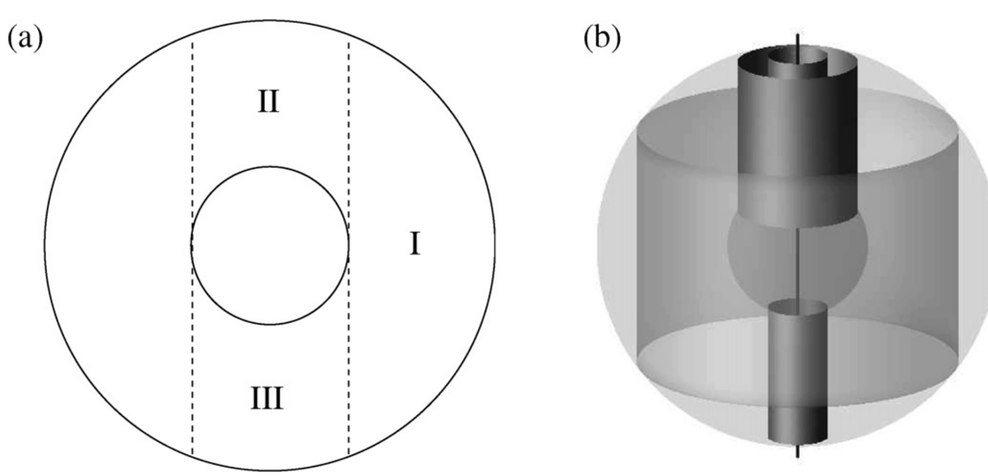

Fig. 1. (a) The three regions in the fluid outer core in which the cylindrical contours, associated with Taylor's constraint, are defined: outside the tangent cylinder (I), inside the tangent cylinder and above (II) and below (III) the inner core. Dashed lines mark the tangent cylinder. (b) Illustrative cylinders over which Taylor's constraint is defined.