Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 56 (2008) 305–310
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Global asymptotic behavior and boundedness of positive solutions to an odd-order rational difference equation[☆]

Maoxin Liao[a,b,*], Xianyi Li[c], Xianhua Tang[a]

[a] School of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410083, PR China
[b] School of Mathematics and Physics, University of South China, Hengyang, Hunan 421001, PR China
[c] School of Mathematics and Computational Science, Shenzhen University, Shenzhen, Guangdong 518060, PR China

Received 27 February 2007; received in revised form 25 November 2007; accepted 4 December 2007

Abstract

In this note we consider the following high-order rational difference equation

$$x_n = 1 + \frac{\prod_{i=1}^{k}(1 - x_{n-i})}{\sum_{i=1}^{k} x_{n-i}}, \quad n = 0, 1, \ldots,$$

where $k \geq 3$ is odd number, $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_{-1}$ is positive numbers. We obtain the boundedness of positive solutions for the above equation, and with the perturbation of initial values, we mainly use the transformation method to prove that the positive equilibrium point of this equation is globally asymptotically stable.

Keywords: Odd-order rational difference equation; Positive equilibrium point; Global asymptotic stability; Boundedness; Transformation method

1. Introduction

Recently, there has been an increasing interest in the study of the global asymptotic properties of rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. As we know, it is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [1–9] for examples to illustrate this. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

☆ Supported partly by NNSF of China (Grant: 10771094) and Project of Hunan Provincial Education Department (Grant: 07C639).
* Corresponding author at: School of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410083, PR China.
E-mail addresses: maoxinliao@163.com, mxliao@163.com (M. Liao).

0898-1221/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2007.12.004

The authors [6] proved that the positive equilibrium point of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + 1}{x_n + x_{n-1}}, \quad n = 0, 1, 2, \ldots \tag{LZ1}$$

with positive initial values x_{-1}, x_0 is globally asymptotically stable.

In fact, Eq. (LZ1) may be rewritten into

$$x_{n+1} = 1 + \frac{(1 - x_n)(1 - x_{n-1})}{x_n + x_{n-1}}, \quad n = 0, 1, \ldots. \tag{LZ2}$$

Motivated by this kind of form of the above Eq. (LZ2), the first author of this paper studied global asymptotic stability for positive solutions to the equation

$$x_{n+1} = 1 + \frac{(1 - x_n)(1 - x_{n-1})(1 - x_{n-2})}{x_n + x_{n-1} + x_{n-2}}, \quad n = 0, 1, 2, \ldots, \tag{L}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are positive numbers and at least two of them are not larger than one.

In this paper, we consider the following high-order rational difference equation

$$x_n = 1 + \frac{\prod_{i=1}^{k}(1 - x_{n-i})}{\sum_{i=1}^{k} x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1}$$

where $k \geq 3$ is odd number, and the initial values $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_{-1} \in [\alpha, \beta]$, with $0 < \alpha < 1$ and $\beta > 1$.

It is clear that the equilibrium $\overline{x}$ of Eq. (1) satisfies

$$\overline{x} = \frac{(1 - \overline{x})^k + k\overline{x}}{k\overline{x}},$$

from which we can get that Eq. (1) has a unique positive equilibrium $\overline{x} = 1$.

Eq. (1) is interesting in its own right. To the best of our knowledge, however, Eq. (1) has not been investigated so far. Therefore, to study its qualitative properties is theoretically meaningful.

It is worthwhile to note that the global asymptotic stability for Eqs. (LZ2) and (L) is proved in [6] via the analysis of semi-cycle structure (similar methods are also used in [8,9]). Such analysis while computationally feasible for small k, can be very involved for larger values. One can see that it is difficult to depict semi-cycle structure for larger k in Eq. (1). It is fortunate that, in this note, the transformation method used does not require prior determination of detailed semi-cycle structure.

The paper proceeds as follows. In Section 2, we obtain the boundedness of positive solutions for Eq. (1), while by introducing some preliminary lemmas and notation, in Section 3, we get a proof of global asymptotic stability for the solutions of Eq. (1) with the perturbation of initial values.

The results obtained in this paper partly generalize the corresponding ones given in paper [6].

2. Boundedness of positive solutions to Eq. (1)

Theorem 2.1. *Suppose that there exist $\alpha \in (0, 1)$ and $\beta \in (1, +\infty)$, such that*

$$\frac{(1 - \alpha)^p (1 - \beta)^q}{p\alpha + q\beta} \leq \beta - 1, \quad p + q = k, q = 0, 2, 4, \ldots, k - 1 \tag{2}$$

and

$$\frac{(1 - \alpha)^p (1 - \beta)^q}{p\alpha + q\beta} \geq \alpha - 1, \quad p + q = k, q = 1, 3, 5, \ldots, k, \tag{3}$$

then the solution $\{x_n\}$ of Eq. (1) satisfies $x_n \in [\alpha, \beta]$, for $n = 0, 1, 2, \ldots$.