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The onset of buoyancy-driven convection in fluid layers with temperature-dependent viscosity

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Abstract

A theoretical analysis of thermal instability driven by buoyancy forces under a time-dependent temperature field of conduction is conducted in an initially quiescent, horizontal liquid layer. The dependency of viscosity on temperature is considered and the propagation theory is employed for the stability analysis. For large Prandtl number systems, the critical condition of the onset of buoyancy-driven convection is obtained as a function of the Rayleigh number and also the viscosity contrast and it is compared with available experimental data. It is evident that the growth period is required until the growing instabilities are detected experimentally, which is dependent upon the viscosity contrast.

Keywords: Buoyancy-driven instability; Temperature-dependent viscosity; Time-dependent cooling; Linear stability theory; Propagation theory

1. Introduction

In a number of systems, such as geological and crystal growth systems, buoyancy-driven convection occurs in fluids whose viscosity is strongly dependent on temperature (Davaille and Jaupart, 1993). For example, in the Earth's mantle and magma chamber temperature differences bring the viscosity variations of up to 10^{30} (Yuen et al., 1981). Since most of real situations are related with rapid cooling or heating, it is important to predict when or where the buoyancy-driven motion sets in.

For initially quiescent, horizontal fluid systems subjected to sudden cooling from above or heating from below, it is well known that buoyancy-driven convection can occur in the transient domain. Their basic temperature profile of conduction is non-linear and also time-dependent. To analyze the related thermal instability quasi-static and transient models have been introduced. In the quasi-static or frozen-time model (Morton, 1957), it is assumed that the growth rate of disturbances is neglected. This model is valid at the large time where the basic temperature field becomes nearly linear. Foster (1965) proposed transient analysis called the amplification theory where the time dependency is treated as an initial value problem. The amplification theory is quite popular, but it involves difficulties in choosing both the initial conditions and also the amplification factor to mark the detection of manifest convection.

For fluid systems whose viscosity is strongly dependent on temperature, the frozen-time model (Yuen et al., 1981) and the amplification theory (Yuen and Fleitout, 1984; Jaupart and Parsons, 1985) have been applied. In the present study, the thermal instability of initially quiescent, horizontal fluid layer cooled from

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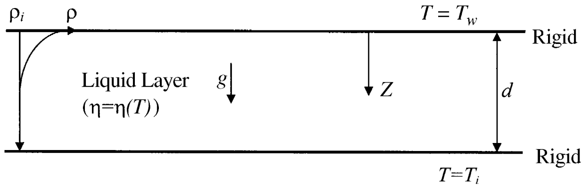

Fig. 1. Schematic diagram of the system considered here.

above is analyzed by taking into account temperature-dependent viscosity variations. Recently, Zaranek and Parmentier (2004), Huang et al. (2003) and Korenaga and Jordan (2003, 2004) analyzed the onset of convection in strongly temperature-dependent viscosity fluids numerically. Zaranek and Parmentier (2004) and Huang et al. (2003) results basically support the scaling theory of Korenaga and Jordan (2003). Here, we will use the propagation theory, which employs linear theory and involves time dependency of disturbances implicitly. This model has been applied to transient Rayleigh–Bénard problems (Kim et al., 2004; Choi et al., 2004a,b), wherein the thermal penetration depth is used as a proper length-scaling factor and the disturbance equations are transformed similarly.

2. Theoretical analysis

2.1. Governing equations

The system considered here is an initially quiescent, horizontal fluid layer of depth d, as shown in Fig. 1. Before cooling the fluid layer is maintained at a uniform temperature T_{i} for time $t<0$. For time $t\geq 0$ the layer is cooled isothermally from above with a constant temperature T_{w}. Employing the Boussinesq approximation, the governing equations of flow and temperature fields are expressed (Davaille and Jaupart, 1993):

$$\nabla\cdot\mathbf{U}=0, \tag{1}$$

$$\rho_{\mathrm{i}}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\nabla\right)\mathbf{U}=-\nabla P+\nabla\cdot(\eta\nabla\mathbf{U}+\eta\nabla\mathbf{U}^{\mathrm{T}})+\rho g, \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\nabla\right)T=\alpha\nabla^2 T, \tag{3}$$

$$\rho=\rho_{\mathrm{i}}[1-\beta(T-T_{\mathrm{i}})], \tag{4}$$

where $\mathbf{U}$ is the velocity vector, P the pressure, η the viscosity function, T the temperature, g the gravitational acceleration, α the thermal diffusivity, ρ the density and β is the thermal expansion coefficient. The subscript "i" represents the reference initial state. The temperature dependency of viscosity is approximated by employing the Frank–Kamenetskii approximation as

$$\eta=\mu_{\mathrm{i}}\exp\{-Q(T-T_{\mathrm{i}})\}, \tag{5}$$

where μ_{i} is the viscosity at the initial temperature and Q is the constant. This functional form represents experimental data of silicone oil quite well (Davaille and Jaupart, 1993), which have been used widely in theoretical analysis (Korenaga and Jordan, 2003; Choblet and Sotin, 2000).

The important parameters to describe the present system are the Prandtl number Pr, the Rayleigh number Ra and the viscosity contrast σ, defined by

$$Pr=\frac{\upsilon_{\mathrm{i}}}{\alpha},\qquad Ra=\frac{g\beta\Delta T d^3}{\alpha\upsilon_{\mathrm{i}}},\qquad \sigma=\frac{\eta(T_{\mathrm{w}})}{\eta(T_{\mathrm{i}})}, \tag{6}$$

where υ_{i} and ΔT denote the kinematic viscosity at initial temperature and temperature difference $(T_{\mathrm{i}}-T_{\mathrm{w}})$, respectively. In case of very slow cooling the basic temperature profile finally becomes linear and time-independent, wherein the critical conditions are well summarized in Chen and Pearlstein (1988) and Selak and Lebon (1993).

For a rapid cooling system of large Ra, the stability problem becomes transient and the critical time t_c to mark the onset of buoyancy-driven motion remains clouded. For this transient stability analysis we define a set of non-dimensionalized variables τ, z and θ_0 by using the scale of time d^2/α, length d and temperature ΔT. Then the basic conduction profile is represented as

$$\theta_0=z-1+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi z)}{n}\exp(-n^2\pi^2\tau), \tag{7a}$$

or

$$\theta_0=\sum_{n=1}^{\infty}\left\{\mathrm{erfc}\left(\frac{n+1}{\sqrt{\tau}}-\frac{\zeta}{2}\right)-\mathrm{erfc}\left(\frac{n}{\sqrt{\tau}}+\frac{\zeta}{2}\right)\right\}, \tag{7b}$$

where $\zeta=z/\sqrt{\tau}$. For deep-pool systems of small time, the following Leveque-type solution is obtained by using the Laplace transform:

$$\theta_0=-\mathrm{erfc}\left(\frac{\zeta}{2}\right). \tag{8}$$

The above equation is in good agreement with the exact solution (7) for $\tau<0.01$.