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An application of symmetry groups to nonlocal continuum mechanics

Teoman Özer

Division of Mechanics, Faculty of Civil Engineering, Istanbul Technical University, 34469 Maslak, Istanbul, Turkey

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Abstract

In this study the symmetry group properties of the one-dimensional elastodynamics problem in nonlocal continuum mechanics is discussed by using an approach developed for symmetry group analysis of integro-differential equations with general form. This approach is based on the modification of the invariance criterion of the differential equations, which include nonlocal variables and integro-differential operators. Lie point symmetries of the nonlocal elasticity equation are obtained based on solving nonlocal determining equations by using a new approach. The symmetry groups for different types of kernel function and the free term including the classical linear elasticity case are presented.

Keywords: Lie symmetry groups; Nonlocal continuum mechanics; Integro-differential equations; Point symmetries; Classification

1. Introduction

This study concerns the symmetry group analysis of the one-dimensional elastodynamics problem of nonlocal elasticity based on a new approach developed for the equations with nonlocal structure. As is well-known, the classical Lie approach that deals with the symmetry group analysis of differential equations has been applied to many problems in mathematical physics and engineering and it is one of the most influential methods used in the study of differential equations. In particular, integration of the system of ordinary differential equations by quadratures, determination of conservations laws by using Noether's theorem, investigation of the reduced forms and similarity solutions of partial differential equations, and linearization of nonlinear ordinary and partial differential equations are some of the specific application fields of Lie symmetry groups. The theoretical concept and main methods of classical Lie theory for the ordinary and partial differential equations are mainly discussed and developed in the works by Sophus Lie [1], Ovsiannikov [2], Olver [3], Bluman [4], Ibragimov [5], and other authors.

In the case of the integro-differential equations, however, it is essential to point out that the classical Lie theory should be modified since the application of this approach fails for these type equations. The nonlocal equations describing the activity of synaptically coupled neuronal networks [6,7], Vlasov–Maxwell equations in plasma physics [8], equations of nonlocal elasticity theory introduced by Eringen [9], the nonlocal equations of the theory of waves

E-mail address: tozer@itu.edu.tr.

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[10], and other classes of the functional-differential equations [11,12] of mathematical physics are very important examples of the problems with integro-differential equations in the literature. In fact, the main difficulty of applying the classical approach to the integro-differential equations comes from their nonlocal form. In the literature, some methods are introduced to deal with this difficulty in symmetry group analysis of equations with nonlocal structure. The method of moments applied in the study [13] by Krasnoslobodtsev and the method of boundary differential equations given by Chetverikov [14] and Chetverikov et al. [15] are the methods referred to as *indirect methods*. The direct methods for investigating solution of nonlocal determining equations are introduced by Bobylev [16], Meleshko [17], Ibragimov, et al. [18], and Özer [19] using Lie–Bäcklund type operators. In addition, there exist also *direct methods* in which a Lie–Bäcklund type operator is not used, but instead of this a Lie point group is used. Such direct methods are introduced by Roberts [20], Özer [23], Akhiev and Özer [21] and Zawistowski [22].

In the present work, a direct method which does not use the Lie–Bäcklund type group transformations but different from studies done by Bobylev, Meleshko, Ibragimov, and Özer, is presented for the analysis of symmetry groups of equations of nonlocal elasticity. Instead of the Lie–Bäcklund type generator, it uses a new definition of Lie point symmetry groups for nonlocal equations with general form [21]. In this definition, nonlocal variables of the equation are also considered as independent variables of a suitable jet space [1–5]. This is similar to the classical theory in which the derivatives of dependent variables with respect to independent variables are considered independent variables. The determining equation obtained by our approach always has one fewer differential order (with respect to the dependent variables of the original equation) than the determining equation obtained by Lie–Bäcklund type operator. Hence, the determination of Lie point symmetries by this approach requires essentially little calculation. The determining equations belonging to the differential equations with nonlocal structure, in general, include also some additional nonlocal variables that are different from nonlocal variables of the original equation. These nonlocal variables determine the extension of a point symmetry group on the nonlocal variables of the original equation and make difficult the solving of the corresponding determining equation. Therefore, a method for the solving of similar determining equations is also presented in the study. The main idea of this method is explained for the one-dimensional nonlocal elasticity equation in the form of a system of integro-differential equations. In fact, the similar system is investigated by Meleshko [17] and Özer [19] by using the Lie–Bäcklund operator (evolutionary vector field) for only special forms of the corresponding equations. However, in this study, the presented method allows us to consider this system of integro-differential equations for more general cases of the kernel function and the free term. We also present some important characteristics of the method that makes it easier to obtain the invariance criterion in the form of nonlocal determining equations than the invariance criterion based on the Lie–Bäcklund type operator. We present symmetry group classification due to the kernel function and the free term.

2. The invariant condition for the functions given by a definite integral

In this section we will introduce the general concept of the invariance criterion for the functions given in the form of a definite integral. First, from the Lie group analysis for the differential equations it is well known that the differentiable function $F(x)$ is called the *invariant function* if and only if $F(x)$ is *invariant* under the Lie group of transformations with infinitesimal generator G [1–5]

$$\tilde{x} = x + a\xi(x) + O(a^2), \qquad G^{(n)}(x; a) = \sum_{k=1}^{n} \xi_k(x)\partial_{x_k} \tag{2.1}$$

in which a is a group parameter, and $x = (x_1, \dots, x_n)$ of R^n and $\xi(x) = (\xi_1(x), \dots, \xi_n(x))$ are infinitesimals (or infinitesimal functions) of the infinitesimal generator G. For the general definition of the invariant condition for the functions given by a definite integral, for our purpose, the following notations are employed for the independent variables x:

$$x_{(q)} = (x_1, \dots, x_q), \qquad x_{(n-q)} = (x_{q+1}, \dots, x_n), \quad 1 \le q \le n \tag{2.2}$$

so that points $x_{(q)}$ and $x_{(n-q)}$ are defined in R^n and R^{n-q}, respectively. Then, we can employ the notation for the point x represented by $x = (x_{(q)}, x_{(n-q)})$. Similarly, for the infinitesimals of the generator G, the notations $\xi_{(q)}(x) = (\xi_1(x), \dots, \xi_q(x))$ and $\xi_{(n-q)}(x) = (\xi_{q+1}(x), \dots, \xi_n(x))$ will be considered, and then we write the notation $\xi(x) = (\xi_{(q)}(x), \xi_{(n-q)}(x))$ for infinitesimals based on new notation introduced for the independent variables x.