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Lower bounds for the Quadratic Minimum Spanning Tree Problem based on reduced cost computation

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ABSTRACT

The Minimum Spanning Tree Problem (MSTP) is one of the most known combinatorial optimization problems. It concerns the determination of a minimum edge-cost subgraph spanning all the vertices of a given connected graph. The Quadratic Minimum Spanning Tree Problem (QMSTP) is a variant of the MSTP whose cost considers also the interaction between every pair of edges of the tree. In this paper we review different strategies found in the literature to compute a lower bound for the QMSTP and develop new bounds based on a reformulation scheme and some new mixed 0–1 linear formulations that result from a reformulation–linearization technique (RLT). The new bounds take advantage of an efficient way to retrieve dual information from the MSTP reduced cost computation. We compare the new bounds with the other bounding procedures in terms of both overall strength and computational effort. Computational experiments indicate that the dual-ascent procedure applied to the new RLT formulation provides the best bounds at the price of increased computational effort, while the bound obtained using the reformulation scheme seems to reasonably tradeoff between the bound tightness and computational effort.

1. Introduction

Given an undirected graph $G=(V,E)$, with $V=\{v_1,v_2,\dots,v_n\}$ and $E=\{e_1,e_2,\dots,e_m\}$, a matrix of quadratic costs C with $C_{ef}\geq 0, \forall e,f\in E$, and linear costs $d_e\geq 0, \forall e\in E$, the quadratic minimum spanning tree problem (QMSTP) consists of finding a spanning tree $T\subseteq G$ with minimum overall cost $\sum_{e,f\in T}C_{ef} + \sum_{e\in T}d_e$.

The QMSTP has been used to model many applications arising in transportation, telecommunication, and energy networks, where linear costs account for the use or construction of edges while the quadratic costs represent the interference between the edges [1,2]. When the interference refers only to pairs of adjacent edges the problem is named adjacent QMSTP (AQMSTP). Both the general QMSTP and the AQMSTP are NP-hard as proved in [1].

Many exact and heuristic algorithms have been proposed for solving both the QMSTP and the AQMSTP. Assad and Xu in [1] proposed a lower bounding procedure and two heuristic approaches. They also described the branch-and-bound algorithm based on a linearized formulation. Öncan and Punnen [3] introduced a Lagrangian relaxation procedure to obtain an improved lower bound and an efficient local search algorithm. Cordone and Passeri [4] have developed two heuristics and an exact approach. Lee and Leung [5] studied the Boolean quadric forest polytope and proposed several facet defining inequalities. Buchheim and Klein [6] proposed complete polyhedral descriptions of the QMSTP with one quadratic term and provide an improved version of the standard linearization by means of cutting planes. Pereira et al. [7] proposed some new formulations using a particular partitioning of the spanning trees, and provided a new mixed binary formulation for the problem by applying the first level of the reformulation-linearization technique (RLT). The most effective heuristic approaches for the QMSTP can be found in [2,8–10].

Lower bounds constitute a fundamental component of branch-and-bound algorithms, and are a basic tool for the evaluation of the quality of heuristic solutions. There are several branch-and bounds for the QMSTP in the literature [1,4,7]. In practice, the lack of efficiently computable tight lower bounds can be one of the main causes of the difficulty of solving even small size instances. The first lower bounding procedure for the QMSTP, proposed by Assad and Xu in [1] iteratively applies an adaptation of the Gilmore–Lawler procedure, originally proposed for the Quadratic Assignment Problem [11,12], to a sequence of equivalent QMSTPs.

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Öncan and Punnen in [3] introduced an extended formulation based on the addition of two sets of valid inequalities to the linearized formulation of [1]. Their lower bounding approach applies a Lagrangian relaxation where the Gilmore–Lawler procedure is used to solve the resulting Lagrangian subproblem. Pereira et al. [7] proposed a new mixed binary formulation for the problem and developed a Lagrangian relaxation approach to obtain a linear programming based lower bound.

The main contribution of this paper consists of obtaining lower bounds for the QMSTP using dual information retrieved from the MSTP reduced costs. In order to find a common framework for description of the individual bounds we review and analyze different bounding procedures proposed in the literature and compare them in terms of continuous relaxation of a Mixed Integer Linear Programming problem (MILP). We describe new bounds for the problem by considering a reformulation of the problem based on dual information retrieved from the continuous relaxation of the MILP. The basic idea is to solve the continuous relaxation of the given MILP, and use the reduced costs as the objective coefficients of the reformulated problem. Moreover, for generating tighter bounds, we develop a mixed 0–1 linear formulation based on the second level of the RLT and show how to handle it via a Lagrangian relaxation where the Lagrangian function has block-diagonal structure. Since the dualized constraints are, indeed, many more than those of the level-1 RLT [7], finding the near-optimal dual multipliers using classical subgradient methods is not viable. Therefore, using the dual information retrieved from the MSTP reduced cost, we devise an efficient dual-ascent procedure to solve the continuous relaxation of the level-2 RLT.

The paper is organized as follows: In Section 2 we review existing lower bounds for QMSTP. In Section 3 we develop a bounding procedure based on a new reformulation scheme. In Section 4 we provide a brief discussion on the level-1 RLT, develop the level-2 RLT representation of QMSTP, and describe our dual-ascent implementation of the level-2 RLT lower bound calculation. In Section 5 we present computational experiments conducted on different benchmarks from the literature and compare the tightness and the required computational time of the new bounding techniques with those of the literature.

2. Problem formulation and lower bounds review

In order to present the mathematical formulation of the QMSTP, let us first introduce some notation used in the sequel. We denote by $E(S)$ the set of all edges with both endpoints in S for any $S \subset V$, and $\delta(i)$ as the set of all edges incident in node i. We define the binary variable x_e to indicate the presence of edge $e \in E$ in the optimal spanning tree. The QMSTP has the following integer formulation:

$$\text{QMSTP}: \quad \min \sum_{e,f \in E} C_{ef} x_e x_f + \sum_{e \in E} d_e x_e$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = n-1 \tag{1}$$

$$\sum_{e \in E(S)} x_e \le |S|-1 \qquad \forall \varnothing \ne S \subset V \tag{2}$$

$$x_e \ge 0, x \text{ binary} \qquad \forall e \in E. \tag{3}$$

where the objective function considers the linear cost of the selected edges and also the interaction costs between pairs of edges. Constraints (2) are the subtour elimination constraints and ensure that no subgraph induced by the nonempty subset $S \subset V$ contains a cycle. These subtour elimination constraints together with the cardinality constraint (1) guarantee the connectivity of the induced subgraph. Constraints (1)–(3) define the set of spanning trees in G and thereafter is denoted by $\mathcal{X}$, i.e.,

$$\mathcal{X} = \{x \ge 0 : (1), (2)\}.$$

2.1. Gilmore–Lawler type bound

The Gilmore–Lawler procedure, shortly denoted by GL, is one of the most popular approaches to find a lower bound for the Quadratic Assignment Problem (QAP). The GL procedure was proposed by Gilmore [11] and Lawler [12] in the context of QAP and has been adapted to many other quadratic 0–1 problems [13,14].

For each edge e, potentially in the solution, we consider the best cumulation cost providing the minimum interaction cost with e. Let P_e be such a subproblem for a given edge $e \in E$

$$P_e: \quad z_e = \min\left\{ \sum_{f \in E} C_{ef} x_f : x \in \mathcal{X}, x_e = 1 \right\} \quad \forall e \in E. \tag{4}$$

The value z_e is the best quadratic contribution to the QMSTP objective function where edge e is in the solution. Once z_e has been computed for each $e \in E$, the GL type bound is given by the solution of the following MSTP:

$$LB_{GL} = \min\left\{ \sum_{e \in E} (z_e + d_e) x_e : x \in \mathcal{X} \right\}. \tag{5}$$

Although the GL bound that we just described is a pure combinatorial bound of the QMSTP, it can also be obtained as the result of a linear programming problem. More precisely, consider the following MILP formulation where the decision variables y_{ef} equal to 1 if and only if both edges e and f are present in the solution of the problem

$$\text{P}: \quad \min \sum_{e,f \in E} C_{ef} y_{ef} + \sum_{e \in E} d_e x_e$$

$$\text{s.t.} \quad \sum_{f \in E} y_{ef} = (n-1) x_e \qquad \forall e \in E \tag{6}$$

$$\sum_{f \in E(S)} y_{ef} \le (|S|-1) x_e \qquad \forall \varnothing \ne S \subset V, e \in E \tag{7}$$

$$y_{ee} = x_e \qquad \forall e \in E \tag{8}$$

$$y_{ef} \ge 0 \qquad \forall e, f \in E$$

$$x \in \mathcal{X}, x \text{ binary}. \tag{9}$$

Constraints (6) guarantee that whenever an edge $e \in E$ is selected, the total number of selected edges interacting with e must be equal to $(n-1)$, including e itself. Overall, constraints (6)–(9) enforce vector $\boldsymbol{y}_e = (y_{e_1 e}, \dots, y_{e_m e})$ to be a spanning tree containing edge e in case $x_e = 1$,or to be the null vector, in case $x_e = 0$.

Consider the continuous relaxation of problem P (CP). The problem CP is computationally interesting since its optimal objective value gives the GL bound as stated in the following theorem.

Theorem 1. *The optimal objective value of CP is equal to LB_{GL}.*

Proof. Let λ, μ_S, α_e, γ_{Se}, and π_e denote the dual variables corresponding to constraints (1), (2), and (6)–(8), respectively. The dual of CP is

$$\text{DCP}: \quad \max \quad -\lambda(n-1) - \sum_{S \subset V} \mu_S(|S|-1) \tag{10}$$

$$\text{s.t.} \quad -\lambda - \sum_{\substack{S \subset V: \\ e \in E(S)}} \mu_S + (n-1)\alpha_e + \sum_{S \subset V} (|S|-1)\gamma_{Se} + \pi_e \le d_e \quad \forall e \in E \tag{11}$$