Computers & Operations Research 52 (2014) 105–112

Contents lists available at ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/caor

On lower bounds for the fixed charge problem

Veena Adlakha [a,*], Krzysztof Kowalski [b]

[a] University of Baltimore, 1420 N. Charles St., Baltimore, MD 21201, USA
[b] State of Connecticut, 121 Cimarron Road, Middletown, CT 06457, USA

ARTICLE INFO

Available online 28 July 2014

Keywords:
Transportation problem
Fixed charge
Quadratic approximation
Branching method

ABSTRACT

In this paper, we develop a quadratic lower bound for the global solution of a fixed charge transportation problem (FCTP). The procedures developed in the paper can be extended to general fixed charge problems and may be incorporated in any branch-and-bound or approximation method to enhance convergence to the optimal solution. We demonstrate the effectiveness of the quadratic lower bound for degenerate FCTPs and suggest ways to improve the bound for large non-degenerate FCTPs by modifying the objective function to extract some variable and fixed charges. A comparative study demonstrates the effectiveness of the quadratic lower bound as compared to the square-root lower bound.

1. Introduction

In transportation problems (TP), the cost of transportation is directly proportional to the number of units transported. In many real-world problems, however, in addition to transportation costs, a fixed cost, sometimes called a setup cost, is also incurred when a distribution variable assumes a non-zero value. Such problems are called fixed-charge transportation problems (FCTP). The optimization literature abounds with different methods for solving a FCTP. The authors of these methods have different claims of computational success in generating an optimal solution. Only two of these methods guarantee an optimal solution: the stage-ranking method [13,16] and the branch-and-bound method [15,17]. Both of these methods are based on enumeration procedures and on a comparison of objective function values for a specified domain of distributions. The exact method of ranking extreme points requires analyzing a large domain of load distributions.

Exact branch-and-bound methods are applicable to small problems only, since the effort required to solve an FCTP using these methods grows exponentially with the size of the problem. Several authors [3,4,7–9,18] have turned to efficient heuristic algorithms for solving FCTPs, because the above-mentioned methods are constrained by limits on computer time. Classical branch-and-bound methods have been applied to several specific real-world applications by other authors [6,11,14,19]. Adlakha et al. [2] developed an analytical method that starts with a linear formulation of the problem and converges to an optimal solution by sequentially separating the fixed costs, and by finding a direction to improve the value of the linear formulation while continually tightening the lower and upper bounds. Adlakha et al. [1] proposed an approximation based on a square-root formulation. However, the square-root formulation and approximation has limitations for large problems.

In this paper, we present a lower bound for the optimal solution of an FCTP by using quadratic approximations of the objective function. We present FCTP formulation in Section 2 and reiterate the linear approximation developed by Balinski [5]. In Section 3, we propose a quadratic approximation leading to a lower bound to an FCTP. The lower bound developed here can be used in tandem with any established algorithm to obtain more effective initial or starting conditions. We start Section 4 with an example of using the quadratic lower bounds with the branching algorithm proposed in Adlakha et al. [2] and continue to carry out computational experiments to study the percentage errors of the optimal solutions as compared to the quadratic lower bounds and the corresponding FCTP values. We illustrate the robustness of the quadratic lower bound as applied to a degenerate FCTP in Section 5. In Section 6, we present improvements in the quadratic bound by modifying the objective function. A comparison with the square-root approximation [1] is presented in Section 7, followed by the conclusions in Section 8.

2. Fixed charge transportation problem

Assume that there are m ($i=1, 2,\dots, m$) suppliers and n ($j=1,2, \dots, n$) customers in a transportation problem. Each supplier i has a_i units of supply, and each customer j has a demand for b_j units. Let x_{ij} be the number of units shipped by supplier i to customer j at a shipping cost per unit c_{ij} plus a fixed cost f_{ij}, incurred for opening

* Corresponding author. Tel.: +1 410 837 4969; fax: +1 410 837 5675.
E-mail address: vadlakha@ubalt.edu (V. Adlakha).

http://dx.doi.org/10.1016/j.cor.2014.07.009
0305-0548/

this route. The objective is to minimize the total cost of meeting all demands, given all supply constraints. The fixed charge transportation problem (FCTP) is formulated as follows:

$$\mathbf{P}: \text{ Minimize } \quad Z = \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}x_{ij} + f_{ij}y_{ij}) \tag{1}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i \quad \forall\ i = 1, 2, ..., m, \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \forall j = 1, 2, ..., n, \tag{3}$$

where

$$y_{ij} = \begin{cases} 1 & \text{if } x_{ij} > 0 \\ 0 & \text{if } x_{ij} = 0 \end{cases} \tag{4}$$

$a_i, b_j, c_{ij}, f_{ij} \geq 0;\ x_{ij} \geq 0 \quad \forall\ (i, j).$

2.1. A linear formulation of the FCTP

Balinski [5] provided a linear approximation of FCTP by relaxing the integer restriction on y_{ij}, with the property that

$$y_{ij} = x_{ij}/m_{ij} \tag{5}$$

So, the relaxed transportation problem of an FCTP would simply be a classical TP with unit transportation costs $C_{ij} = c_{ij} + f_{ij}/m_{ij}$. We refer to this problem as **PB**. The optimal solution $\{x_{ij}^B\}$ to problem **PB** can be easily modified into a feasible solution of $\{x_{ij}^B, y_{ij}^B\}$ of **P** by setting $y_{ij}^B = 1$ if x_{ij}^B is positive and $y_{ij}^B = 0$ otherwise. Balinski shows that the optimal value, $Z(\mathbf{PB})$, provides a lower bound on the optimal value $Z^*(\mathbf{P})$ of FCTP, i.e., $Z(\mathbf{PB}) = \sum\sum C_{ij}x_{ij}^B \leq Z^*(\mathbf{P})$. Geometrically, the Balinski linear approximation can be represented as in Fig. 1.

3. A quadratic approximation to lower bound

In this section, we develop a quadratic approximation, Q_L, for FCTP costs as proposed in Fig. 2. The objective here is to estimate FCTP costs more closely than the linear approximation used by Balinski [5], in order to improve the solution estimate for the FCTP.

Define

$$Q_L(x_{ij}) = \alpha(x_{ij})^2 + \beta(x_{ij}) + \varepsilon, \tag{6}$$

where

$$Q_L(0) = 0, \tag{7}$$

$$Q_L(m_{ij}) = f_{ij} + c_{ij}m_{ij}, \tag{8}$$

$$Q_L'(m_{ij}) = c_{ij}. \tag{9}$$

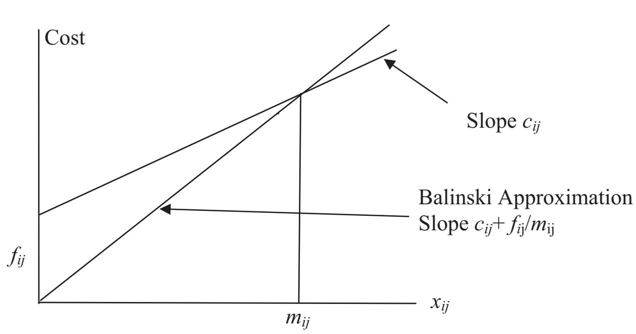

Fig. 1. Balinski linear approximation.

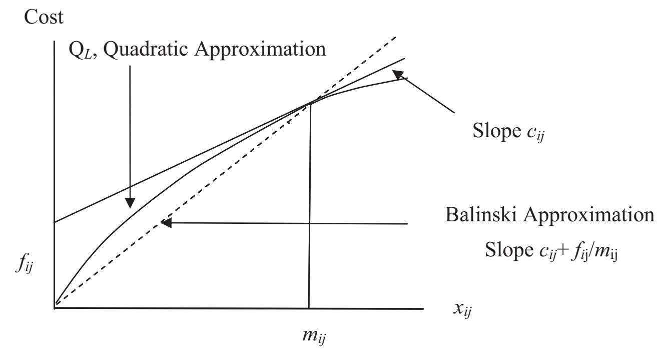

Fig. 2. Quadratic approximation of FCTP.

Eqs. (7) and (8) ensure that the proposed quadratic curve, $Q_L(x_{ij})$ starts at the origin and equals the FCTP cost at m_{ij}. Eq. (9), where Q_L' denotes the derivative of Q_L, ensures that $Q_L(x_{ij})$ is tangential to FCTP at m_{ij}. We use these equations to determine coefficients in $Q_L(x_{ij})$.

$Q_L(0) = 0 \Rightarrow \varepsilon = 0.$

Eqs. (8) and (9) yield

$$\alpha(m_{ij})^2 + \beta(m_{ij}) \quad = f_{ij} + m_{ij}c_{ij}$$

and

$$2\alpha(m_{ij}) + \beta \quad = c_{ij}$$

Solving these two above equations, we get

$$\alpha = -f_{ij}/(m_{ij})^2 \text{ and } \beta = 2f_{ij}/(m_{ij}) + c_{ij}$$

Therefore

$$Q_L(x_{ij}) = -\{f_{ij}/(m_{ij})^2\}(x_{ij})^2 + \{(2f_{ij}/m_{ij}) + c_{ij}\}(x_{ij}).$$

Theorem 1. *The quadratic curve, $Q_L(x_{ij})$, lies entirely above the Balinski approximation and entirely below the FCTP cost line.*

Proof. We first show that $\mathbf{PB}(x_{ij}) \leq Q_L(x_{ij})$ for all $0 \leq x_{ij} \leq m_{ij}$.

Consider

$$\begin{aligned} Q_L(x_{ij}) - \mathbf{PB}(x_{ij}) &= -\{f_{ij}/(m_{ij})^2\}(x_{ij})^2 + \{(2f_{ij}/m_{ij}) + c_{ij}\}x_{ij} - \{(f_{ij}/m_{ij}) + c_{ij}\}x_{ij} \\ &= -\{f_{ij}/(m_{ij})^2\}(x_{ij})^2 + (f_{ij}/m_{ij})x_{ij} \\ &= f_{ij}\{(x_{ij}/m_{ij}) - (x_{ij}/m_{ij})^2\} \\ &= f_{ij}(x_{ij}/m_{ij})\{1 - (x_{ij}/m_{ij})\} \\ &\geq 0 \quad \text{for } 0 \leq x_{ij} \leq m_{ij} \text{ and } f_{ij} \geq 0. \end{aligned}$$

Now consider $Q_L(x_{ij})$ versus $P(x_{ij}) = (c_{ij}x_{ij} + f_{ij}y_{ij})$ as defined in problem **P**. It is clear that $Q_L(x_{ij}) = P(x_{ij}) = 0$ as $y_{ij} = 0$ when $x_{ij} = 0$. Assume $x_{ij} > 0$ so that $y_{ij} = 1$.

$$\begin{aligned} P(x_{ij}) - Q_L(x_{ij}) &= (c_{ij}x_{ij} + f_{ij}) - \{(2f_{ij}/m_{ij}) + c_{ij}\}x_{ij} + \{f_{ij}/(m_{ij})^2\}(x_{ij})^2 \\ &= f_{ij}\{1 - 2(x_{ij}/m_{ij}) + (x_{ij}/m_{ij})^2\} \\ &= f_{ij}\{1 - (x_{ij}/m_{ij})\}^2 \end{aligned}$$

Since $0 \leq \{1 - (x_{ij}/m_{ij})\}^2 \leq 1$ for $0 \leq x_{ij} \leq m_{ij}$ and $f_{ij} \geq 0$, the theorem follows. □

Consider the following quadratic problem:

$$\mathbf{PQ}_L: \text{ Minimize}$$

$$Z = \sum_{i=1}^{m}\sum_{j=1}^{n} -\{f_{ij}/(m_{ij})^2\}(x_{ij})^2 + \{(2f_{ij}/m_{ij}) + c_{ij}\}x_{ij} \tag{10}$$