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# An exact algorithm and a metaheuristic for the generalized vehicle routing problem with flexible fleet size

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ABSTRACT

The *generalized vehicle routing problem* (GVRP) involves finding a minimum-length set of vehicle routes passing through a set of clusters, where each cluster contains a number of vertices, such that the tour includes exactly one vertex from each cluster and satisfies capacity constraints. We consider a version of the GVRP where the number of vehicles is a decision variable. This paper introduces a new mathematical formulation based on a two-commodity flow model. We solve the problem using a branch-and-cut algorithm and a metaheuristic that is a hybrid of the *greedy randomized adaptive search procedure* (GRASP) and the *evolutionary local search* (ELS) proposed in [18]. We perform computational experiments on instances from the literature to demonstrate the performance of our algorithms.



## 1. Introduction

The *capacitated vehicle routing problem* (CVRP) is one of the most popular and challenging combinatorial optimization problems. It involves finding the optimal set of routes for a fleet of vehicles that serves a given set of customers. In classical transportation problems, each customer is served from only one vertex. Therefore, there is always a well-defined set of vertices that must be visited, and we need to find the solution from this set. However, in many real applications a customer can be served from more than one vertex, and the resulting problems are more complex. The GVRP is a generalization of the CVRP and also an extension of the *generalized traveling salesman problem* (GTSP). The GVRP can model problems concerned with the design of bilevel transportation networks; see [6] and [16] for further information on its applications.

The GVRP is defined as follows. Let $G=(V,E)$ be an undirected graph, where $V$ is the vertex set and $E$ is the edge set. $V=\{v_0,\dots,v_{n-1}\}$ is the set of $n$ vertices that can be visited, and vertex $v_0$ is the depot, containing $m$ identical vehicles with a common capacity $Q$. $C=\{C_0,\dots,C_{K-1}\}$ is the set of $K$ clusters. Each cluster $C_i$ except $C_0$, which contains only the depot, has a demand $D_i$. Each cluster includes a number of vertices of $V$, and every vertex in $V$ belongs to exactly one cluster. For each $v_i \in V$, let $\alpha(i)$ be the cluster that contains vertex $v_i$. The term $D(S)=\sum_{i|C_i\subseteq S} D_i$ is used to represent the total demand in set $S$ which is a subset of $V$. The number of vehicles $m$ can be constant or variable. A length $c_{ij}$ is associated with each edge of $E=\{\{v_i,v_j\}: v_i,v_j\in V, i<j\}$. The GVRP consists in finding $m$ vehicle routes such that (i) each route begins and ends at the depot; (ii) each route visits exactly one vertex of each cluster and visits it only once; (iii) the demand served by each route does not exceed the vehicle capacity $Q$; and (iv) the total cost is minimized.

The GVRP is clearly NP-hard since it reduces to a VRP when each cluster includes only one vertex or to a GTSP when the capacity constraints are relaxed. The number of papers on this topic is quite limited. The problem was first introduced by Ghiani and Improta [8]. In 2003, Kara and Bektaş [9] proposed the first formulation that was polynomial in the number of constraints and variables. An ant colony algorithm and a genetic algorithm were proposed in [17] and [14] respectively. Recently, Bektaş et al. [6] proposed four formulations and four branch-and-cut algorithms. They concluded that the best formulation was an undirected two-index flow one based on an exponential number of constraints. They also proposed a heuristic based on *large neighborhood search* (LNS) to provide upper bounds for the branch-and-cut algorithms. At the same time, Pop et al. [16] introduced two new formulations. The first, the node formulation, is similar to the formulation in [9] but produces a stronger lower bound, and the second is a flow-based formulation. The authors directly solved one instance from [8] using CPLEX. They reported no further computational experience with the proposed formulations and did not develop branch-and-cut algorithms.

In this paper, we consider a version of the GVRP that has not been investigated in the literature where the number of vehicles is

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a decision variable. This version of the problem would allow determining the appropriate fleet size to minimize the daily routing cost. We make two contributions: (i) we present a new formulation for the GVRP, and (ii) we propose an exact method and a metaheuristic to solve the problem. Computational experiments show that our exact approach can solve instances of up to 121 vertices and 51 clusters, and our metaheuristic gives high-quality solutions for the instances tested in a reasonable computational time. Compared to the formulation proposed in [6], our formulation provides better lower bounds and better performance of the branch-and-cut algorithm.

The remainder of the paper is organized as follows. Section 2 describes our formulation and several valid inequalities. The branch-and-cut algorithm and the metaheuristic are presented in Sections 3 and 4 respectively. Section 5 discusses the computational results, and Section 6 summarizes our conclusions.

## 2. New formulation for the GVRP

We first reintroduce the best formulation of [6]. Note that Bektaş et al. [6] tackle the case in which the number of vehicles is constant. To adapt it to the context where the number of vehicles is variable, we simply consider the number of vehicles $m$ in the formulation as a decision variable. This formulation uses integer variables $z_{ij}, \{v_i, v_j\} \in E$ that count the number of times the edge $\{v_i, v_j\}$ is used. Let $\delta(S) = \{\{v_i, v_j\} \in E : v_i \in S, v_j \notin S\}$ and $z(F) = \sum_{\{v_i,v_j\} \in F} z_{ij}$ where $F$ is a subset of $E$. Then the formulation is as follows:

$$\text{Minimize} \quad \sum_{\{v_i,v_j\} \in E} c_{ij} x_{ij} \tag{1}$$

$$\text{Subject to} \quad z(\delta(C_k)) = 2 \quad \forall C_k \in C \backslash C_0 \tag{2}$$

$$z(\delta(C_0)) = 2m \tag{3}$$

$$z(\delta(S)) + 2 \sum_{\{v_i,v_j\} \in L : i \notin S} z(\{i\} : C_j) \le 2 \quad \forall C_k \in C \backslash C_0, \ S \subseteq C_k, \ L \in \overline{L}_k \tag{4}$$

$$\sum_{(v_i \in S_1, v_j \in S_2)} z_{ij} \le |S| - \lceil \frac{D(S)}{Q} \rceil \quad \forall S_1 \subseteq S, \ S_2 \subseteq S, S \subseteq C, |S| \ge 2 \tag{5}$$

$$z_{ij} \in \{0, 1, 2\} \quad \forall \{v_i, v_j\} \in \delta(0) \tag{6}$$

$$z_{ij} \in \{0, 1\} \quad \forall \{v_i, v_j\} \in E \backslash \delta(0) \tag{7}$$

$$m \in \mathbb{N}. \tag{8}$$

In this formulation, constraints (2) ensure that each cluster is visited exactly once. Constraints (3) imply that $m$ vehicles will leave the depot. Constraints (4) are referred to as same-vertex inequalities; they ensure that when a vehicle arrives at a certain vertex in a cluster, it will depart from the same vertex. Here, $\overline{L}_k = \{L : L \subseteq \bigcup_{i \in C_k} L_i, \ |L \cap L_i| = 1, \ \forall i \in C_k\}$ where $L_i = \{i\} \times (C \backslash \{0, \alpha(i)\})$, defined for all $v_i \in V \backslash v_0$. Constraints (5) are the capacity constraints.

We now describe a new integer programming formulation for the GVRP. The idea underlying this formulation was first introduced by [7] for the traveling salesman problem (TSP). Langevin et al. [11] extended this approach to the TSP with time windows. Baldacci et al. [3] used it to derive a new formulation and a branch-and-cut algorithm for the VRP, and Baldacci et al. [2] adapted it for the *covering tour problem* (CTP) without capacity constraints. Currently, together with the two-index flow formulation and the set partitioning formulation, this is one of the most successful formulations underlying exact methods for the CVRP (see [4]).

Our formulation is an extension of that proposed by Baldacci et al. [3] for the CVRP. To adapt this idea for the GVRP, we assume that each vertex $v_i$ of $V$ has a demand $d_i$ equal to the demand $D_{\alpha(i)}$ of the cluster to which it belongs. In other words, all the vertices in a cluster have the same demand as that of the cluster. The difference from CVRP is that we do not need to visit all the vertices of $V$.

We first extend the original graph $G$ to $\overline{G} = (\overline{V}, \overline{E})$ by adding a new vertex $v_n$, which is a copy of the depot $v_0$. We now have $\overline{V} = V \cup \{v_n\}$, $V' = \overline{V} \backslash \{v_0, v_n\}$, $\overline{E} = E \cup \{\{v_i, v_n\}, v_i \in V'\}$, and $c_{in} = c_{0i} \forall v_i \in V'$.

This formulation requires two flow variables, $f_{ij}$ and $f_{ji}$, to represent an edge of a feasible GVRP solution along which the vehicle carries a load of $Q$ units. When the vehicle travels from $v_i$ to $v_j$, flow $f_{ij}$ represents the load collected and flow $f_{ji}$ represents the empty space of the vehicle (i.e., $f_{ji} = Q - f_{ij}$).

Let $x_{ij}$ be a 0–1 variable equal to 1 if edge $\{v_i, v_j\}$ is used in the solution and 0 otherwise. Let $y_i$ be a binary variable that indicates the use of vertex $v_i$ in the solution. Then the GVRP can be stated as

$$\text{Minimize} \quad \sum_{\{v_i,v_j\} \in \overline{E}} c_{ij} x_{ij} \tag{9}$$

$$\text{Subject to} \quad \sum_{v_i \in C_k} y_i = 1 \quad \forall C_k \in C \tag{10}$$

$$\sum_{v_i \in \overline{V}, i < k} x_{ik} + \sum_{v_j \in \overline{V}, j > k} x_{kj} = 2y_k \quad \forall v_k \in V' \tag{11}$$

$$\sum_{v_j \in \overline{V}} (f_{ji} - f_{ij}) = 2d_i y_i \quad \forall v_i \in V' \tag{12}$$

$$\sum_{v_j \in V'} f_{0j} = \sum_{v_i \in V'} d_i y_i \tag{13}$$

$$\sum_{j \in V'} f_{nj} = mQ \tag{14}$$

$$f_{ij} + f_{ji} = Q x_{ij} \quad \forall \{v_i, v_j\} \in \overline{E} \tag{15}$$

$$f_{ij} \ge 0, f_{ji} \ge 0 \quad \forall \{v_i, v_j\} \in \overline{E} \tag{16}$$

$$x_{ij} \in \{0, 1\} \quad \forall \{v_i, v_j\} \in \overline{E} \tag{17}$$

$$y_i \in \{0, 1\} \quad \forall v_i \in V' \tag{18}$$

$$m \in \mathbb{N}. \tag{19}$$

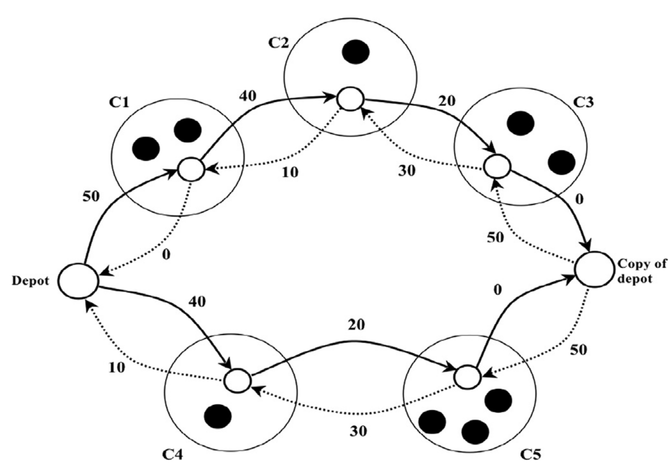


**Fig. 1.** Flow paths for solution with two routes.