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# Generalized multiple depot traveling salesmen problem—Polyhedral study and exact algorithm

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ABSTRACT

The generalized multiple depot traveling salesmen problem (GMDTSP) is a variant of the multiple depot traveling salesmen problem (MDTSP), where each salesman starts at a distinct depot, the targets are partitioned into clusters and at least one target in each cluster is visited by some salesman. The GMDTSP is an NP-hard problem as it generalizes the MDTSP and has practical applications in design of ring networks, vehicle routing, flexible manufacturing, scheduling and postal routing. We present an integer programming formulation for the GMDTSP and valid inequalities to strengthen the linear programming relaxation. Furthermore, we present a polyhedral analysis of the convex hull of feasible solutions to the GMDTSP and derive facet-defining inequalities that strengthen the linear programming relaxation of the GMDTSP. All these results are then used to develop a branch-and-cut algorithm to obtain optimal solutions to the problem. The performance of the algorithm is evaluated through extensive computational experiments on several benchmark instances.



## 1. Introduction

The generalized multiple depot travelling salesmen problem (GMDTSP) is an important combinatorial optimization problem that has several practical applications including, but not limited to, maritime transportation, health-care logistics, survivable telecommunication network design [3], material flow system design, postbox collection [19], and routing unmanned vehicles [20,23]. The GMDTSP is formally defined as follows: let $D:=\{d_1,\dots,d_k\}$ denote the set of depots and $T$, the set of targets. We are given a complete undirected graph $G=(V,E)$ with vertex set $V:=T\cup D$ and edge set $E:=\{(i,j): i\in V, j\in T\}$. In addition, a proper partition $C_1,\dots,C_m$ of $T$ is given; these partitions are called *clusters*. For each edge $(i,j)=e\in E$, we associate a non-negative cost $c_e=c_{ij}$. The GMDTSP consists of determining a set of at most $k$ simple cycles such that each cycle starts and ends at a distinct depot, at least one target from each cluster is visited by some cycle and the total cost of the set of cycles is a minimum. The GMDTSP reduces to a multiple depot traveling salesmen problem (MDTSP – [5]) when every cluster is a singleton set. The GMDTSP involves two related decisions:

1. choosing a subset of targets $S\subseteq T$, such that $|S\cap C_h|\geq 1$ for $h=1,\dots,m$;
2. solving a MDTSP on the subgraph of $G$ induced by $S\cup D$.

The GMDTSP can be considered either as a generalization of the MDTSP in Benavent and Martínez [5] where the targets are partitioned into clusters and at least one target in each cluster has to be visited by some salesman or as a multiple salesmen variant of the symmetric generalized traveling salesman problem (GTSP) in Fischetti et al. [8,9]. Benavent and Martínez [5] and Fischetti et al. [8] present a polyhedral study of the MDTSP and GTSP polytope respectively, and develop a branch-and-cut algorithm to compute optimal solutions for the respective problem.

This is the first work in the literature that analyzes the facial structure and derives additional valid and facet-defining inequalities for the convex hull of feasible solutions to the GMDTSP. This paper presents an integer linear programming formulation and develops a branch-and-cut algorithm to solve the problem to optimality. This work generalizes the results of the two aforementioned problems namely, the MDTSP [5] and the GTSP [8].

### 1.1. Related work

A special case of the GMDTSP with one salesman, the symmetric generalized traveling salesman problem (GTSP), was first introduced by Labordere [15] and Srivastava et al. [26] in relation

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to record balancing problems arising in computer design and to the routing of clients through agencies providing various services, respectively. Since then, the GTSP has attracted considerable attention in the literature as several variants of the classical traveling salesman problem can be modeled as a GTSP [19,7,22,20]. Noon and Bean [21] developed a procedure to transform a GTSP to an asymmetric traveling salesman problem and Laporte et al. [18] investigated the asymmetric counterpart of the GTSP. Despite most of the aforementioned applications of the GTSP [19] extending naturally to their multiple depot variant, there are no exact algorithms in the literature to address the GMDTSP.

A related generalization of the GMDTSP can be found in the vehicle routing problem (VRP) literature. VRPs are capacitated counterparts for the TSPs where the vehicles have a limited capacity and each target is associated with a demand that has to be met by the vehicle visiting that target. The multiple VRPs can be classified based on whether the vehicles start from a single depot or from multiple depots. The generalized multiple vehicle routing problem (GVRP) is a capacitated version of the GMDTSP with all the vehicles starting from a single depot. Bektas et al. [3] present four formulations for the GVRP, compare their linear relaxation solutions, and develop a branch-and-cut to optimally solve the problem. In [17], Laporte models the GVRP as a location-routing problem. On the contrary, Ghiani and Improta [10] develop an algorithm to transform the GVRP into a capacitated arc routing problem, which therefore enables one to utilize the available algorithms for the latter to solve the former. In a more recent paper, Bautista et al. [2] study a special case of the GVRP derived from a waste collection application where each cluster contains at most two vertices. The authors describe a number of heuristic solution procedures, including two constructive heuristics, a local search method and an ant colony heuristic to solve several practical instances. To our knowledge, there are no algorithms in the literature to compute optimal solutions to the generalized multiple depot vehicle routing problem or the GMDTSP.

The objective of this paper is to develop an integer programming formulation for the GMDTSP, study the facial structure of the GMDTSP polytope and develop a branch-and-cut algorithm to solve the problem to optimality. The rest of the paper is organized as follows: in Section 2, we introduce notation and present the integer programming formulation. In Section 3, the facial structure of the GMDTSP polytope is studied and its relation to the MDTSP polytope [5] is established. We also introduce a general theorem that allows one to lift any facet of the MDTSP polytope into a facet of the GMDTSP polytope. We further use this result to develop several classes of facet-defining inequalities for the GMDTSP. In the subsequent sections, the formulation is used to develop a branch-and-cut algorithm to obtain optimal solutions. The performance of the algorithm is evaluated through extensive computational experiments on 116 benchmark instances from the GTSP library [12].

## 2. Problem formulation

We now present a mathematical formulation for the GMDTSP inspired by models in [5] and [8]. We propose a two-index formulation for the GMDTSP. We associate to each feasible solution $\mathcal{F}$, a vector $\mathbf{x} \in \mathbb{R}^{|E|}$ (a real vector indexed by the elements of $E$) such that the value of the component $x_e$ associated with edge $e$ is the number of times $e$ appears in the feasible solution $\mathcal{F}$. Note that for some edges $e \in E$, $x_e \in \{0,1,2\}$ *i.e.*, we allow the degenerate case where a cycle can only consist of a depot and a target. If $e$ connects two vertices $i$ and $j$, then $(i,j)$ and $e$ will be used interchangeably to denote the same edge. Similarly, associated to $\mathcal{F}$, is also a vector $\mathbf{y} \in \mathbb{R}^{|T|}$, *i.e.*,a real vector indexed by the elements of $T$. The value of the component $y_i$ associated with a target $i \in T$ is equal to one if the target $i$ is visited by a cycle and zero otherwise.

For any $S \subset V$, we define $\gamma(S) = \{(i,j) \in E : i,j \in S\}$ and $\delta(S) = \{(i,j) \in E : i \in S,\ j \notin S\}$. If $S = \{i\}$, we simply write $\delta(i)$ instead of $\delta(\{i\})$. We also denote by $C_{h(v)}$ the cluster containing the target $v$ and define $W := \{v \in T : |C_{h(v)}| = 1\}$. Finally, for any $\hat{E} \subseteq E$, we define $x(\overline{E}) = \sum_{(i,j) \in \overline{E}} x_{ij}$, and for any disjoint subsets $A, B \subseteq V$, $(A:B) = \{(i,j) \in E : i \in A, j \in B\}$ and $x(A:B) = \sum_{e \in (A:B)} x_e$, Using the above notations, the GMDTSP is formulated as a mixed integer linear program as follows:

$$\text{Minimize} \quad \sum_{e \in E} c_e x_e \tag{1}$$

subject to

$$x(\delta(i)) = 2y_i \quad \forall i \in T, \tag{2}$$

$$\sum_{i \in C_h} y_i \geq 1 \quad \forall h \in \{1,\ldots,m\}, \tag{3}$$

$$x(\delta(S)) \geq 2y_i \quad \forall S \subseteq T, i \in S, \tag{4}$$

$$x(D':\{j\}) + 3x_{jk} + x(\{k\}:D \setminus D') \leq 2(y_j + y_k) \quad \forall j,k \in T; D' \subset D, \tag{5}$$

$$x(D':\{j\}) + 2x(\gamma(S \cup \{j,k\})) + x(\{k\}:D \setminus D') \leq \sum_{v \in S} 2\,y_v + 2(y_j + y_k) - y_i$$
$$\forall i \in S; j,k \in T, j \neq k; S \subseteq T \setminus \{j,k\}, S \neq \emptyset; D' \subset D, \tag{6}$$

$$x_e \in \{0,1\} \quad \forall e \in \gamma(T), \tag{7}$$

$$x_e \in \{0,1,2\} \quad \forall e \in (D:T), \tag{8}$$

$$y_i \in \{0,1\} \quad \forall i \in T. \tag{9}$$

In the above formulation, the constraints (2) ensure the that number of edges incident on any vertex $i \in T$ is equal to 2 if and only if target $i$ is visited by a cycle ($y_i = 1$). The constraints (3) force at least one target in each cluster to be visited. The constraints (4) are the connectivity or sub-tour elimination constraints. They ensure a feasible solution that has no sub-tours of any subset of customers in $T$. The constraints (5) and (6) are the path elimination constraints. They do not allow for any cycle in a feasible solution to consist of more than one depot. The validity of these constraints is discussed in Section 2.1. Finally, the constraints (7)–(9) are the integrality restrictions on the $\mathbf{x}$ and $\mathbf{y}$ vectors.

### 2.1. Path elimination constraints

The first version of the path elimination constraints was developed in the context of location routing problems by Laporte et al. [16]. Laporte et al. named these constraints as chain-barring constraints. The authors in [4] and [5] use similar path elimination constraints for the location routing and the multiple depot traveling salesmen problems. The version of path elimination constraints used in this paper is adapted from [27]. Any path that originates from a depot and visits exactly two customers before terminating at another depot is removed by the constraint (5). The validity of the constraint (5) can be easily verified [16] and [27]. Any other path $d_1, t_1, \ldots, t_p, d_2$, where $d_1, d_2 \in D$, $t_1, \ldots, t_p \in T$ and $p \geq 3$, violates inequality (6) with $D' = \{d_1\}$, $S = \{t_2, \ldots, t_{p-1}\}$, $j = t_1$, $k = t_p$ and $i = t_r$ where $2 \leq r \leq p-1$. The proof of validity of the constraint (6) is discussed as a part of the polyhedral analysis of the polytope of feasible solutions to the GMDTSP in the next section (see Proposition 3.5).

We note that our formulation allows for a feasible solution with paths connecting two depots and visiting exactly one customer. We refer to such paths as 2-paths. As the formulation allows for two copies of an edge between a depot and a target, 2-paths can