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Unsplittable non-additive capacitated network design using set functions polyhedra

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we address the *Unsplittable Non-Additive Capacitated Network Design* problem, a variant of the Capacitated Network Design problem where the flow of each commodity cannot be split, even between two facilities installed on the same link. We propose a compact formulation and an aggregated formulation for the problem. The latter requires additional inequalities from considering each individual arc-set. Instead of studying those particular polyhedra, we consider a much more general object, the unitary step monotonically increasing set function polyhedra, and identify some families of facets. The inequalities that are obtained by specializing those facets to the Bin Packing function are separated in a Branch-and-Cut for the problem. Several series of experiments are conducted on random and realistic instances to give an insight on the efficiency of the introduced valid inequalities.

1. Introduction

The design of optimal networks has become one of the major economic issues for nowadays telecommunications industry. Many variants of this problem have been considered in the literature, addressing the topology aspects as well as the installation of capacities and the traffic routing. One of the network design problems that has received a big attention is the so-called capacitated network design (CND) problem. Given a network with a set of commodities and a set of potential capacitated link facilities together with their costs, the problem consists of determining the facilities to install on the network so that the commodities can be routed and the total cost is minimum.

In this paper, we consider a variant of the CND problem. This concerns the case where the commodities cannot be split. More precisely, for its routing, each commodity must go from its origin to its destination through only one path and must use at most one facility on each link of the network. The latter constraint makes impossible to aggregate the capacities installed over a link, and will be referred to as the *non-additivity* of the facilities. This problem arises in the design of telecommunication networks. In particular, we are interested in optical networks holding a set of multiplexer devices interconnected by optical fibres and using the so-called OFDM (*Orthogonal Frequency Division Multiplexing*) technology. Indeed, this technology consists in setting up several facilities referred to as *subbands* on the links of a network. Every subband has a certain capacity and a non-negative cost. In this context, given an optical network, a set of commodities and a set of available subbands, the aim is to identify the minimum cost subbands to install on the links of the network so that the traffic may be routed. In particular, we focus on the problem which concerns the installation of the subbands, which will permit an optimal routing. In fact, an efficient algorithm for solving this restricted version of the problem, which is already NP-hard, as it will be shown later, may be useful for solving the problem of the more general multilayer version. This is our motivation for considering the problem which will be called the *Unsplittable Non-Additive Capacitated Network Design* (UNACND) problem.

The purpose of this paper is to devise a Branch-and-Cut algorithm for the UNACND problem. The algorithm is based on an investigation of the polyhedral structure of the problem when it is restricted to a single link. Previous works have already shown the effectiveness of such approach for solving network design problems (see [1–3] and the references therein). Some results in this paper are presented in a very preliminary stage in [4].

To the best of our knowledge, the UNACND problem has not been considered before. However, other versions of the problem have been widely discussed in the literature. In fact, the restriction of CND to one arc has been investigated first by Magnanti et al. [5], for two facilities and splittable flow assumption. Pochet and Wolsey [6] study the polyhedron of a single-arc network design problem with an arbitrary number of facilities and splittable flow assumption. Brockmüller et al. [1] and van Hoesel et al. [2] investigate the

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CND restricted to one edge (the edge capacity problem). They study the integer knapsack problem arising from this relaxation and introduce the so-called c-strong inequalities and give necessary and sufficient conditions for these inequalities to define facets. In [2], the authors give conditions under which the facets of edge capacity polytope also define facets for the CND polytope. In [3], Atamtürk and Rajan study both splittable and unsplittable CND arc-set polyhedra by considering the existing capacity of the arc. They give a linear-time separation procedure for the residual capacity inequalities and show its effectiveness for the splittable CND. They also use the c-strong inequalities and derive a second class of valid inequalities for the unsplittable CND problem. Similar approaches have also been used to study cut-set polyhedra associated with the CND in [7] and CND with survivability constraints in [8].

Besides, the earlier results on the CND problem and the associated polyhedron can be found in [5], where the authors study a multiple commodities-two facilities network design problem restricted to a single arc. They propose several classes of facet defining valid inequalities that completely describe the convex hull of the arc-set CND solutions. In [9], Magnanti et al. propose a more detailed discussion on the CND problem. They propose two approaches to solve the problem: a Lagrangian approach and a cutting planes approach. In particular, they show that the results given in [5] strengthen the CND formulation. Some of the results given in [5] are generalized by Bienstock and Günlük in [10]. They are also extended for the capacity expansion problem, where the overall capacity of the network can be increased by installing several units of capacitated facilities or "batches" on the links. The authors develop a cutting plane approach based on several facet defining inequalities, to solve the problem. Further polyhedral results are presented in [11–16] for different versions of the CND problem under splittable traffic assumption. In particular in [15,12], the authors study two formulations based on the so-called metric inequalities for the minimum cost CND problem. In [12], Bienstock et al. describe two classes of valid inequalities that define facets and are used to obtain a complete characterization of the considered polyhedron for complete three nodes graphs. Moreover, Mattia et al. [15] introduce the so-called tight metric inequalities and show that all the facets of the polyhedron associated with the solutions of the CND are tight metric inequalities. Note that handling the problem by this approach is similar to the Benders decomposition approach (see [17] for more details on this approach).

More recently, some authors have turned their attention to the multi-layer version of the CND problem (see for instance [18–20] and the references therein). Most of the approaches proposed to solve the multi-layer network design problems are based on the results introduced for their single-layer versions.

Our contribution. The objective of this paper is to solve efficiently the UNACND problem by using a Branch-and-Cut algorithm that embeds new classes of valid inequalities. These are obtained by investigating the polyhedra associated with the single arc UNACND problem. Actually, we realized that different possible variants of the single arc CND are in fact associated with the same polyhedron. We refer to these variants as *functions*. We then introduce the polyhedra associated with a general class of functions called *unitary step monotonically increasing* functions, and study their basic properties. We provide two classes of inequalities called *Min Set I* and *Min Set II* that are valid for all considered functions. We give necessary and sufficient conditions for these inequalities to define facets. Our polyhedral results as well as the separation routines remain available for every considered function, by integrating the specificities of each function. We give an application to the *Bin Packing function*, that is in fact equivalent to the arc-set UNACND. In particular, our results for Min Set I inequalities generalize those provided in [1–3] for c-strong inequalities. Both classes of inequalities Min Set I and Min Set II are used within a Branch-and-Cut algorithm to efficiently solve UNACND problem and to strengthen the linear relaxation of the multi-layer version of this problem.

The rest of the paper is organized as follows. In Section 2 we briefly describe the UNACND problem and its restriction to a single arc. In Section 3, we introduce the set functions polyhedra and study their basic properties. We then present the Min Set I and Min Set II inequalities, and investigate their facial structure. In Section 4, we give an application of our polyhedral results to the Bin Packing function, and show the interest of such application for the UNACND problem. Both Min Set I and Min Set II inequalities are embedded within a Branch-and-Cut algorithm described in Section 5. In this section, we also present the separation procedures used to generate the identified valid inequalities. We then show a set of experiments conducted on random and realistic SNDlib based instances in Section 6. Finally, some concluding remarks are given in Section 7.

2. The unsplittable non-additive capacitated network design problem

The UNACND problem can be presented as follows. Consider a bi-directed graph $G=(V,A)$ that represents an optical network. Each node $v \in V$ corresponds to an optical device (multiplexer) and every arc $a=(i,j) \in A$ corresponds to an optical fibre. If an arc (i,j) exists in A, then (j,i) also belongs to A. Let K be a set of commodities. Each commodity $k \in K$ has an origin node $o_k \in V$, a destination node $d_k \in V$ and a traffic $D^k > 0$ that has to be routed between o_k and d_k. We suppose that a set of equivalent modules, each of capacity C, is available. This set will be denoted by W. Assume that $D^k \le C$, for all $k \in K$. A module $w \in W$ installed on an arc (i,j) is a copy of that arc, and yields a cost c_{ij}. Every module w can carry one or many commodities, but a commodity cannot be split on several paths or even on several modules of the same arc. This specificity makes impossible to aggregate the commodities having the same source and destination nodes, to reduce the size of the problem. Thus, there might be several different commodities with the same origin and destination nodes. The UNACND problem is to determine a minimum cost set of modules that have to be installed on the arcs of G so that a routing path is associated with each commodity from its origin to its destination.

Now consider a set $K=\{1,\dots,n\}$ of items (demands) with weights $D^1, D^2, \dots, D^n$ and bins with the same capacity C. The *bin packing problem* (BPP) consists in assigning each item to one bin so that the total weight of the items in each bin does not exceed C and the number of bins used is minimum [21]. We assume, without loss of generality, that the weights D^k and the capacity C are positive integers and $D^k \le C$, for all $k \in K$. The bin packing problem is NP-hard in general [22] and various approaches have been proposed during the last three decades to solve it. In what follows, we use the relationship between UNACND problem and bin packing problem to show that the former is NP-hard.

Proposition 1. *The UNACND problem is NP-hard even if A has a single arc.*

Proof. We will show that the UNACND problem is NP-hard even when the underlying graph consists of only one arc. The reduction is from the bin packing problem. Consider an instance of the bin packing problem, given by a set of items denoted K, each one having a weight $D^k > 0$, $k \in K$. Let W denote a set of available bins, where every bin has a capacity C. We look for the smallest number of bins needed to pack the items of K. Let us construct the graph $G=(V,A)$, where $V=\{u,v\}$ and $A=\{(u,v)\}$. In other words, G consists of two nodes interconnected by a single arc. For each $k \in K$, we must send D^k units of flow from node u to node v. The set W defines the set of available modules with capacity C, the installation costs are unitary. Let B denote the optimal solution of