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Formulations for the nonbifurcated hop-constrained multicommodity capacitated fixed-charge network design problem

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ABSTRACT

This paper addresses the multicommodity capacitated fixed-charge network design problem with nonbifurcated flows and hop constraints. We present and compare mathematical programming formulations for this problem and we study different relaxations: Lagrangean relaxations, linear programming relaxations, and partial relaxations of the integrality constraints. In particular, we show that the Lagrangean bound obtained by relaxing the flow conservation equations is tighter than the linear programming relaxation bound. We present computational results on a large set of randomly generated instances.

1. Introduction

Let $G=(V,E)$ be a directed graph, where V is the set of nodes and E is the set of arcs. Let also K be a set of commodities, where each commodity $k \in K$ is defined by an origin node s^k, a destination node t^k, and a demand d^k to be routed from s^k to t^k. Each arc $e \in E$ has a capacity u_e that satisfies $u_e \le \sum_{k \in K} d^k$. For each unit of commodity k going through arc e, a nonnegative routing cost c_e^k has to be paid. Moreover, a nonnegative design cost f_e applies if there is a positive flow of any commodity on arc e. We consider the *multicommodity capacitated fixed-charge network design problem with nonbifurcated flows and hop constraints* (MCFDH) in which we want to minimize the sum of routing and design costs, while satisfying the demands and the capacity constraints. In addition, each commodity k has to be routed on a single path (nonbifurcated or unsplittable flows) whose length must not exceed l^k. These hop constraints are useful in the context of reliability and quality of service in telecommunication and transportation networks, where limiting the number of arcs can reduce the probability of information loss or avoid unacceptable delays. When $f_e=0$, $e \in E$, and $l^k=|E|$, $k \in K$, the MCFDH reduces to the *multicommodity integral flow problem*, which is NP-hard even if the number of commodities is two [15]. Thus, the MCFDH is itself NP-hard.

Network design problems with bifurcated (or splittable) flows have been well studied (see [16], and the references therein). Problems in which demands cannot be split arise in several applications in the areas of telecommunication and transportation. Brockmüller et al. [7] study a capacitated network design problem with non-linear costs arising in the design of private line networks; a similar problem is treated in Dahl et al. [14]. In Gavish and Altinkemer [17] a non-linear network design problem is studied, while in Balakrishnan et al. [4] the authors present a decomposition algorithm for trees. Barnhart et al. [5] present a column generation model and a branch-and-price-and-cut algorithm for the integer multicommodity flow problem.

Problems involving hop constraints have been studied for minimum spanning tree problems in Gouveia et al. [23], Dahl et al. [13], Gouveia and Requejo [24], Gouveia [18,19], and for Steiner tree problems in Costa et al. [9] and Voß [29], in which both model the design of centralized telecommunication networks with minimum cost, as well as in Balakrishnan and Altinkemer [3] for more general telecommunication network design problems. Survivability in network design problems, which deals with the design of networks that can survive arc or node failures, is investigated in Botton et al. [6], Gouveia et al. [21,22], Alevras et al. [1,2]. The effect of hop limits on the optimal cost is studied in Orlowsky and Wessäly [27] for a telecommunication network design problem. The convex hull of hop-constrained st-paths in a graph is studied in Dahl [11] and Dahl and Gouveia [12], which give a complete linear description when the number of hops is not larger than three, and propose classes of facet-defining inequalities for the general case.

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In this paper, we present three mathematical programming formulations for the MCFDH: the classical arc-based and path-based formulations, as well as the hop-indexed model. Different relaxations of these formulations are studied: Lagrangean relaxations, linear programming (LP) relaxations, and partial relaxations of the integrality constraints. We first focus on the hop-indexed model and study two Lagrangean relaxations, one obtained by relaxing the capacity constraints and the other by relaxing the flow conservation constraints. The first Lagrangean relaxation can be decomposed into $|K|$ hop-constrained shortest path problems. We show that the hop-indexed formulation for that problem has the integrality property, which implies that its associated Lagrangean dual has the same value as the LP relaxation value of the hop-indexed model. The second Lagrangean relaxation can be decomposed into $|E|$ 0–1 knapsack problems, which do not have the integrality property. Thus, the value of the Lagrangean dual associated with this relaxation is greater than, or equal to, the LP relaxation value (this is a major difference with the bifurcated case, where a similar Lagrangean relaxation provides the same bound as the LP relaxation). A theoretical comparison of the LP relaxations of the formulations is then performed, showing that the path-based formulation and the hop-indexed formulation have the same LP relaxation value, which is not worse (and typically better) than the LP relaxation of the arc-based formulation. We also compare these relaxations with those obtained by relaxing the integrality of either the design variables or the flow variables.

The paper is organized as follows. Mathematical programming formulations of the problem are presented in Section 2. In Section 3, we present the Lagrangean relaxations and compare them to the LP relaxations and to the partial relaxations of the integrality constraints. Computational results are presented and analyzed in Section 4. Section 5 concludes the paper.

2. Problem formulations

This section presents three mathematical programming formulations for the MCFDH, namely, the classical arc-based and path-based models, as well as the hop-indexed formulation.

2.1. Classical arc-based formulation

This formulation is obtained by adding the hop constraints to the classical arc-based formulation of the multicommodity capacitated fixed-charge network design problem. It uses binary variables x_e^k taking value 1 if the path of commodity k goes through arc e, and 0 otherwise, as well as binary variables y_e taking value 1 if arc e carries flow for at least one commodity, and 0, otherwise. Given $v \in V$, we denote by $\omega^+(v)$ the set of outgoing arcs from v and by $\omega^-(v)$ the set of incoming arcs to v.

$$(C) \quad \min \sum_{k \in K} \sum_{e \in E} d^k c_e^k x_e^k + \sum_{e \in E} f_e y_e$$

$$\sum_{e \in \omega^+(v)} x_e^k - \sum_{e \in \omega^-(v)} x_e^k = \begin{cases} 1, & v = s^k \\ -1, & v = t^k \\ 0, & v \in V \backslash \{s^k, t^k\} \end{cases}, \quad k \in K \tag{1}$$

$$\sum_{k \in K} d^k x_e^k \le u_e y_e, \quad e \in E \tag{2}$$

$$x_e^k \le y_e, \quad e \in E, \ k \in K \tag{3}$$

$$\sum_{e \in E} x_e^k \le l^k, \quad k \in K \tag{4}$$

$$x_e^k \in \{0,1\}, \quad e \in E, \ k \in K \tag{5}$$

$$y_e \in \{0,1\}, \quad e \in E. \tag{6}$$

Constraints (1) are the flow conservation constraints, while (2) are the capacity constraints. Constraints (3) are redundant strong linking inequalities, which significantly improve the LP relaxation of the model. Inequalities (4) represent the hop constraints, which are valid because the flows are nonbifurcated.

2.2. Path-based formulation

For every $k \in K$, let P^k be the set of paths from s^k to t^k whose length is less than or equal to l^k. The formulation uses binary variables y_e as in the classical arc-based model, as well as binary variables x_p taking value 1 if $p \in P^k$ is used to satisfy the demand for commodity k, and 0 otherwise. Given a path p, we define $a_{ep} = 1$ if arc e belongs to path p, and 0 otherwise. The cost per unit of flow on path $p \in P^k$ is then $c_p = \sum_{e \in E} a_{ep} c_e^k$.

$$(P) \quad \min \sum_{k \in K} \sum_{p \in P^k} d^k c_p x_p + \sum_{e \in E} f_e y_e$$

$$\sum_{p \in P^k} x_p = 1, \quad k \in K \tag{7}$$

$$\sum_{k \in K} \sum_{p \in P^k} a_{ep} d^k x_p \le u_e y_e, \quad e \in E \tag{8}$$

$$\sum_{p \in P^k} a_{ep} x_p \le y_e, \quad e \in E, \ k \in K \tag{9}$$

$$x_p \in \{0,1\}, \quad p \in P^k, \ k \in K \tag{10}$$

$$y_e \in \{0,1\}, \quad e \in E. \tag{11}$$

Constraints (7) ensure that a single path is selected for each commodity. Capacity and strong linking constraints are represented by (8) and (9), respectively. Finally, as a feasible path $p \in P^k$ has a length smaller than or equal to l^k, the hop constraints are satisfied by any solution to this formulation.

2.3. Hop-indexed formulation

For every commodity k, every arc e and every possible position q with $1 \le q \le l^k$, we define variable x_{eq}^k equal to 1 if arc e appears in position q in the path from s^k to t^k and 0, otherwise.

$$(I) \quad \min \sum_{k \in K} \sum_{e \in E} \sum_{q=1}^{l^k} d^k c_e^k x_{eq}^k + \sum_{e \in E} f_e y_e$$

$$\sum_{e \in \omega^+(v)} \sum_{q=1}^{l^k} x_{eq}^k - \sum_{e \in \omega^-(v)} \sum_{q=1}^{l^k} x_{eq}^k = \begin{cases} 1, & v = s^k \\ -1, & v = t^k \end{cases}, \quad k \in K \tag{12}$$

$$\sum_{e \in \omega^+(v)} x_{eq}^k - \sum_{e \in \omega^-(v)} x_{eq-1}^k = 0, \quad k \in K, \ v \in V \backslash \{s^k, t^k\}, \ q = 2, \dots, l^k \tag{13}$$

$$\sum_{k \in K} \sum_{q=1}^{l^k} d^k x_{eq}^k \le u_e y_e, \quad e \in E \tag{14}$$

$$\sum_{q=1}^{l_k} x_{eq}^k \le y_e, \quad e \in E, \ k \in K \tag{15}$$

$$x_{eq}^k \in \{0,1\}, \quad k \in K, \ e \in E, \ q = 1, \dots, l^k \tag{16}$$

$$y_e \in \{0,1\}, \quad e \in E. \tag{17}$$

Constraints (14) and (15) are capacity and strong linking constraints, respectively. Constraints (12) and (13) are the flow conservation constraints at the origin/destination nodes and at the