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A new formulation and approach for the black and white traveling salesman problem

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ABSTRACT

This paper proposes a new formulation and a column generation approach for the black and white traveling salesman problem. This problem is an extension of the traveling salesman problem in which the vertex set is divided into black vertices and white vertices. The number of white vertices visited and the length of the path between two consecutive black vertices are constrained. The objective of this problem is to find the shortest Hamiltonian cycle that covers all vertices satisfying the cardinality and the length constraints. We present a new formulation for the undirected version of this problem, which is amenable to the Dantzig–Wolfe decomposition. The decomposed problem which is defined on a multigraph becomes the traveling salesman problem with an extra constraint set in which the variable set is the feasible paths between pairs of black vertices. In this paper, a column generation algorithm is designed to solve the linear programming relaxation of this problem. The resulting pricing subproblem is an elementary shortest path problem with resource constraints, and we employ acceleration strategies to solve this subproblem effectively. The linear programming relaxation bound is strengthened by a cutting plane procedure, and then column generation is embedded within a branch-and-bound algorithm to compute optimal integer solutions. The proposed algorithm is used to solve randomly generated instances with up to 80 vertices.

1. Introduction

The black and white traveling salesman problem (BWTSP) is an extension of the traveling salesman problem (TSP) in which each vertex is classified as either black or white. As in the TSP, the aim is to find the shortest Hamiltonian cycle. However, in the BWTSP, the number of white vertices visited between any pair of consecutive black vertices is limited by the cardinality constraint, and the length of the path (chain) between any such pair is subject to the length constraint. The BWTSP can be defined on a graph $G=(V,E)$, where $V=B\cup W$ is the set of vertices which is itself composed of a subset of black vertices indexed by $b\in B$ and a subset of white vertices indexed by $w\in W$, and $E=\{(i,j):i,j\in V, i\neq j\}$ is the set of edges. Each edge $(i,j)\in E$ is associated with a distance c_{ij}. In the undirected case of this problem, which is also handled in this paper, (i,j) and (j,i) represent the same edge, and $c_{ij}=c_{ji}$. A feasible solution to this problem is an ordered set of the form $b_1,w_1,\dots,b_2,w_k,\dots,b_{l-1},\dots,b_l$ where $b_1=b_l$, and the cardinality of the path (chain) consisting of white vertices between b_{h-1} and b_h for some $l\geq h>1$ is at most Q due to the cardinality constraint, and the length of it is at most L due to the length constraint. A *chain* between two consecutive black vertices will be referred to as a *path* in order to stick with the common terminology used throughout the paper. The BWTSP is to determine a Hamiltonian cycle with the smallest cost satisfying the cardinality and the length constraints. When $Q=L=\infty$, the BWTSP is equivalent to the TSP (see [12] for the methods and extensions of the TSP).

Two of the applications of the BWTSP are in aircraft scheduling [18] and telecommunications [20,4]. In the former application, the flight legs between a pair of stations correspond to the white vertices, and the maintenance stations are defined as black vertices in a flight network. There exists an arc between a pair of vertices if the vertex pair is operable in sequence. A cost of performing a pair of vertices in sequence is associated with each arc. The directed BWTSP defined on this network is to determine the flight sequences between two maintenance stations that satisfy the flight cardinality and the distance limitations. In the latter application, the aim is to improve the telecommunication reliability through a survivable network (SONET) architecture. The resulting network design problem is to determine a circular sequence of hubs (white vertices) and ring offices (black vertices) in such a way that the number of hubs and the distance between two consecutive ring offices do not exceed preset values.

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There exist both exact and heuristic algorithms to solve the BWTSP. A simulated annealing heuristic for the problem is developed by Mak and Boland [16]. They also consider a method based on the Lagrangian relaxation of constraints including the subtour elimination constraints. A lower bound is obtained by solving the Lagrangian dual problem by modified subgradient optimization. Their proposed heuristic approach finds feasible solutions within 20 min with gaps of less than 3% on instances involving 36 vertices. Bourgeois et al. [2] develop and compare three construction heuristics that were originally proposed for the TSP. A feasibility and an improvement heuristics, based on a 2-opt scheme, are also employed to obtain a good feasible solution. The variety of exact methods to solve this problem is limited. An enumeration-based optimization algorithm is proposed by Wasem [20] to solve problems with tight cardinality limits. Ghiani et al. [10] propose an integer linear programming formulation of the undirected BWTSP. They develop several classes of valid inequalities to strengthen the cardinality and the length constraints, and proposed various procedures, including heuristics, in order to separate these inequalities. The cutting plane procedure is embedded in a branch-and-bound algorithm to find the optimal solution. Their proposed approach solves instances involving up to 100 vertices when $L=\infty$, and its performance deteriorates as the cardinality and the length constraints tighten.

In this paper, we propose a new formulation for the undirected BWTSP which extends the variable set defined in the existing formulation of Ghiani et al. [10]. This new formulation arises from the fact that each edge can be traversed between a single pair of consecutive black vertices, and it is amenable to the Dantzig–Wolfe decomposition [5]. The variable set in the decomposed problem is the set of feasible paths between pairs of black vertices. We model this problem on a multigraph in which the vertex set is composed of the black vertices, and the set of edges between a pair of black vertices corresponds to the feasible paths formed by white vertices, as well as the original edge between them. The aim is still to find the shortest Hamiltonian cycle on black vertices subject to an extra constraint set which partitions the white vertex set into disjoint paths. Since this formulation contains many columns, we design a column generation algorithm to solve its linear programming (LP) relaxation. The resulting pricing subproblem is an elementary shortest path problem with resource constraints (ESPPRC) which is known to be an NP-Hard problem. The label-correcting algorithm developed by Feillet et al. [9] is modified to solve this problem. In order to accelerate the solution of the subproblem, we employ strategies such as devising a column pool, a bounding procedure and a heuristic approach. For the same purpose, we test the ng-path relaxation of the ESPPRC proposed by Baldacci et al. [1] on the BWTSP. The LP relaxation bound of this problem found by the column generation algorithm is improved using the subset-row inequalities, a valid inequality set defined for the set packing polytope developed by Jepsen et al. [14]. The column generation algorithm is embedded within a branch-and-price algorithm to reach the integer optimal solution. Consequently, the contributions of this paper are the new formulation of the BWTSP and its reformulation by the Dantzig–Wolfe decomposition, and a branch-and-price approach which incorporates various strategies in column generation to solve the resulting reformulation. The effectiveness of the proposed approach is constrained by the complexity of the ESPPRC. Nonetheless, we have been able to solve instances with up to 80 vertices. As opposed to the branch-and-cut algorithm proposed by Ghiani et al. [10], the performance of our proposed approach enhances as the constraints tighten, which will be shown through computational experiments.

The paper is organized as follows. In Section 2, a new formulation for the BWTSP is presented together with its Dantzig–Wolfe reformulation. In this section, the details of the column generation algorithm, which includes the ESPPRC and the methods to solve it efficiently, for the LP relaxation of this problem are also explained. The valid inequalities to strengthen the LP relaxation, the initial upper-bounding method and the details of the branch-and-price algorithm are given in Section 3. This is followed by the computational experiments in Section 4, and by conclusions in Section 5.

2. A new formulation and its decomposition

In this section, we first explain a new formulation for the BWTSP. The new formulation is designed to incorporate into the variable definition of Ghiani et al. [10] the constraint that a white vertex can be visited between only a single pair of consecutive black vertices. Hence, each variable x_e, $e \in E$ is decomposed into a set of variables, namely $x_e = \sum_{b_i b_j \in B, b_i \neq b_j} x_e^{b_i b_j}$, where $x_e^{b_i b_j} = 1$ if and only if edge $e \in E$ is traversed between the pair of consecutive black vertices $b_i, b_j \in B, b_i \neq b_j$. The Dantzig–Wolfe decomposition is applied to this new formulation, which leads to a reformulation containing many columns and only a small number of constraints. The pricing subproblem resulting from the application of the column generation is modeled as an ESPPRC, and the label-correcting algorithm and the acceleration techniques for this subproblem are explained in the rest of the section.

The notation necessary in the model definition is given next. For a subset of vertices S, subsets of the edge set E, denoted by $\delta(S)$ and $E(S)$, correspond to the edge set incident to S, $\delta(S) = \{(v_i, v_j) : v_i \in S, v_j \in V \backslash S\}$, and the set of edges within S, $E(S) = \{(v_i, v_j) : v_i, v_j \in S\}$, respectively. For a single vertex $v \in V$, the notations $\delta(v)$ and $E(v)$ are used instead of $\delta(\{v\})$ and $E(\{v\})$, respectively. As defined above, our model consists of a variable for each edge $e \in E$ and for each pair of black vertices $b_i, b_j \in B, b_i \neq b_j$, denoted by $x_e^{b_i b_j}$. This is equivalent to defining an edge between each pair of vertices for each pair of black vertices. This variable definition simplifies the constraints that bound the length and the cardinality of the paths linking consecutive black vertices. Moreover, in order to indicate the assignment of a white vertex w between a pair of consecutive black vertices $b_i, b_j \in B, b_i \neq b_j$, the variable $y_w^{b_i b_j}$ is included in the formulation.

The formulation of the BWTSP is

$$\text{minimize} \quad \sum_{b_i, b_j \in B, b_i \neq b_j} \sum_{e \in E} c_e x_e^{b_i b_j} \tag{1}$$

$$\text{subject to} \quad \sum_{b_i, b_j \in B, b_i \neq b_j} \sum_{e \in \delta(w)} x_e^{b_i b_j} = 2, \quad w \in W, \tag{2}$$

$$\sum_{b_j \in B, b_j \neq b_i} \sum_{e \in \delta(b_i)} x_e^{b_i b_j} = 2, \quad b_i \in B, \tag{3}$$

$$\sum_{e \in \delta(w)} x_e^{b_i b_j} = 2 y_w^{b_i b_j}, \quad w \in W, b_i, b_j \in B, b_i \neq b_j, \tag{4}$$

$$\sum_{e \in E} c_e x_e^{b_i b_j} \leq L, \quad b_i, b_j \in B, b_i \neq b_j, \tag{5}$$

$$\sum_{w \in W} y_w^{b_i b_j} \leq Q, \quad b_i, b_j \in B, b_i \neq b_j, \tag{6}$$

$$\sum_{e \in \delta(S)} x_e^{b_i b_j} \geq 2, \quad b_i, b_j \in B, b_i \neq b_j, S \subseteq W \cup \{b_i, b_j\}, |S| \geq 2, \tag{7}$$

$$\sum_{b_i \in S, b_j \notin S} \sum_{e \in \delta(b_i)} x_e^{b_i b_j} \geq 2, \quad S \subset B, |B|-1 \geq |S| \geq 2, \tag{8}$$

$$x_e^{b_i b_j} \in \{0, 1\}, \quad b_i, b_j \in B, \; b_i \neq b_j, \; e \in E, \tag{9}$$

$$y_w^{b_i b_j} \in \{0, 1\}, \quad b_i, b_j \in B, \; b_i \neq b_j, \; w \in W. \tag{10}$$