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A nonisolated optimal solution of general linear multiplicative programming problems ☆

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ABSTRACT

This article presents a finite branch-and-bound algorithm for globally solving general linear multiplicative programming problems (GLMP). The proposed algorithm is based on the recently developed theory of monotonic optimization. The proposed algorithm provides a nonisolated global optimal solution, and it turns out that such an optimal solution is adequately guaranteed to be feasible and to be close to the actual optimal solution. It can be shown by the numerical results that the proposed algorithm is effective and the computational results can be gained in short time.

1. Introduction

Consider the following general linear multiplicative program:

$$(\mathrm{GLMP})\begin{cases} \min & \Phi(y)=\sum_{i=1}^{t}\prod_{j=1}^{p_i}(c_{ij}^{\mathrm{T}}y+d_{ij}) \\ \text{s.t.} & Ay-b\leqslant 0, \\ & y\geqslant 0, \end{cases}$$

where $c_{ij}\in R^N$ and $d_{ij}\in R$ for $i=1,\ldots,t,\ j=1,\ldots,p_i$ and $A\in R^{m\times N}$, $b\in R^m$. Assume further that the feasible set

$$\Lambda=\{y|Ay-b\leqslant 0, y\geqslant 0\}=\{y|F_j(y)=b_j-a_jy\geqslant 0,\ j=1,\ldots,m,\ y\geqslant 0\}$$

is nonempty bounded polyhedron.

With our broader definition, one can see quadratic programming (QP), bilinear programming (BLP), as well as linear multiplicative program (LMP) fall into the category of (GLMP). The problem (GLMP) has attracted considerable attention in the literature because of its large number of practical applications in various fields of study, including microeconomics [1], financial optimization [2,3], VLSI chip design [4,5], decision tree optimization [6], portfolio optimization [2,7], plant layout design [8], multicriteria optimization problems [9], robust optimization [10], and data mining/pattern recognition [11].

The problem (GLMP) is obviously multiextremal, for its special cases such as (LMP), (BLP), and (QP) are multiextremal. Both (LMP) and (GLMP) are known to be NP-hard problems [12,13].

In the last decade, many solution algorithms have been proposed for globally solving the problem (GLMP). The methods can be classified as parameter-based methods [14–16], branch-and-bound methods [17–21], outer-approximation methods [22,23], mixed branch-and-bound and outer-approximation [24], vertex enumeration methods [25,26], outcome-space cutting plane methods [27], and heuristic methods [28,29].

In this paper we will suggest a finite branch-and-bound algorithm to solve the problem (GLMP) based on the recently developed theory of monotonic optimization [30,31]. The goal of this research is twofold. First, we present a transformation of the problem based on the characteristics of problem (GLMP). Thus the original problem (GLMP) is equivalently reformulated as a monotonic optimization problem (P1). Second, by using a special procedure of monotonic optimization problem (P1), we propose a finite branch-and-bound algorithm for problem (GLMP). This algorithm consists in seeking the best nonisolated feasible solution. This solution, i.e., the nonisolated optimal solution which is computed by the proposed approach is adequately guaranteed to be feasible and to be close to the actual optimal solution. Hence, the proposed approach can find a more appropriate approximate optimal solution which is also stable under small perturbations of the constraints. This stresses the importance of nonisolation for practical implementation of global optimization methods, because the problem of finding feasible and stable solutions is a fundamental question for a global optimization problem.

The remainder of this paper is organized as follows. The next section converts the problem (GLMP) into a monotonic optimization problem. Section 3 introduces the concept of nonisolated optimality.

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In addition, a method for finding such a nonisolated optimal solution is presented in this section. The rectangular branching process, the reducing process and the upper bounding process used in this approach are defined and studied in Section 4. Section 5 incorporates this approach into a branch-and-bound algorithm for solving problem (GLMP), and shows the convergence properties of the proposed algorithm. In Section 6, we give the results of solving some numerical examples with the proposed algorithm.

2. Equivalent reformulation

A function $f : R^n \to R$ is said to be increasing if $f(x') \leqslant f(x)$ for all $x', x \in R^n$ satisfying $x' \leqslant x$, i.e. $x'_i \leqslant x_i$, $\forall i = 1, \ldots, n$. Any function that can be represented as the difference of two increasing functions is said to be a d.m. function.

In this section we show that the problem (GLMP) can be transformed into an equivalent monotonic optimization problem with increasing objective function and d.m. constrained functions.

Theorem 2.1. *Any polynomial $Q(y_1, y_2, \ldots, y_n)$ (in particular any affine or quadratic function) is a d.m. function on any box $[a, b] \subset R^n_+$.*

Proof. By grouping separately the terms with positive coefficients and those with negative coefficients, one can write $Q(y) = Q^+(y) - Q^-(y)$, where each Q^+, Q^- is a polynomial with nonnegative coefficients, hence an increasing function, on $[a, b] \subset R^n_+$. In the case of a linear function $Q(y) = \langle c, y\rangle$ we can write $c = c^+ - c^-$ with $c_i^+ = \max\{0, c_i\}$, $c_i^- = \max\{0, -c_i\}$, so that $Q^+(y) = \langle c^+, y\rangle$, $Q^-(y) = \langle c^-, y\rangle$. □

Note that the objective function of the problem (GLMP) is a polynomial in R^N and the constraint functions $F_j(y)$ $(j=1, \ldots, m)$ are affine. Let $\Phi(y) = \Phi^+(y) - \Phi^-(y)$ and $F_j(y) = F_j^+(y) - F_j^-(y)$ be the d.m. representations of $\Phi(y)$, $F_j(y)$ as described in the problem (GLMP). In addition, by solving some linear programming problems, we can obtain the initial upper bound and lower bound $\bar{y}_i, \underline{y}_i$ of each variable y_i $(i=1, \ldots, N)$ and initial partition rectangle $Y = \{y \mid \underline{y}_i \leqslant y_i \leqslant \bar{y}_i, i=1, \ldots, N\}$ containing the feasible region of problem (GLMP), where

$$\underline{y} = (\underline{y}_i)_{N\times 1} \quad \text{with } \underline{y}_i = \min_{y\in\Lambda} y_i,\ i = 1, \ldots, N,$$

$$\bar{y} = (\bar{y}_i)_{N\times 1} \quad \text{with } \bar{y}_i = \max_{y\in\Lambda} y_i,\ i = 1, \ldots, N.$$

Since $\Phi^-(\underline{y}) \leqslant \Phi^-(y) \leqslant \Phi^-(\bar{y})$, $\forall y \in Y = [\underline{y}, \bar{y}]$, then by introducing an additional variable $z \in R$ to (GLMP), can thus convert (GLMP) into the following equivalent problem:

$$(\text{P1})\begin{cases} \min & \Phi^+(y) + z \\ \text{s.t.} & F_j^+(y) - F_j^-(y) \geqslant 0, \quad j = 1, \ldots, m, \\ & Phi^-(y) + z \geqslant 0, \\ & y \in Y = \{y \mid 0 \leqslant \underline{y}_i \leqslant y_i \leqslant \bar{y}_i, \quad i = 1, \ldots, N\}, \\ & -\Phi^-(\bar{y}) \leqslant z \leqslant -\Phi^-(\underline{y}). \end{cases}$$

Clearly the objective function of (P1) is increasing and each constrained function is a d.m. function. To globally solve problem (GLMP), the branch-and-bound algorithm will globally solve problem (P1). The validity of this approach follows from the following result.

Theorem 2.2. *If (y^*, z^*) is a global optimal solution for the problem* (P1), *then y^* is a global optimal solution for the problem* (GLMP). *Conversely, if y^* is a global optimal solution for the problem* (GLMP), *then (y^*, z^*) is a global optimal solution for the problem* (P1), *where, $z^* = -\Phi^-(y^*)$.*

Proof. The proof of this theorem follows easily from the definitions of problems (GLMP) and (P1), therefore, it is omitted. □

In addition, for the sake of simplicity, we let $x = (y, z) \in R^{N+1}$ with $y \in R^N$, $z \in R$ and let $n = N + 1$. Then, without loss of generality, by changing the notation, the problem (P1) can be rewritten as the form:

$$(\text{P}) \quad \min\{g(x) \mid h(x) \geqslant 0,\ x \in X_0 = [x^l, x^u]\},$$

where

$$X_0 = \{x \in R^n \mid x_i^l \leqslant x_i \leqslant x_i^u,\ i = 1, \ldots, n\} = \left\{ x \in R^n \,\middle|\, \begin{array}{ll} \underline{y}_i \leqslant x_i = y_i \leqslant \bar{y}_i, & i = 1, \ldots, N, \\ -\Phi^-(\bar{y}) \leqslant x_i = z \leqslant -\Phi^-(\underline{y}), & i = N + 1 \end{array} \right\},$$

and

$$g(x) = \Phi^+(y) + z \tag{1}$$

is an increasing function, while

$$h(x) = \min_{k=1,\ldots,k_0} \{u_k(x) - v_k(x)\}, \quad k_0 = m + 1, \tag{2}$$

with $u_k(x)$, $v_k(x)$ being increasing functions such that

$$u_k(x) = \begin{cases} F_k^+(y) & \text{if } k = 1, \ldots, m, \\ \Phi^-(y) + z & \text{if } k = m + 1, \end{cases} \tag{3}$$

and

$$v_k(x) = \begin{cases} F_k^-(y) & \text{if } k = 1, \ldots, m, \\ 0 & \text{if } k = m + 1. \end{cases} \tag{4}$$

Based on the above discussion, here, from now on we assume that the original problem (GLMP) has been equivalently converted to the problem (P), with $g(x)$ increasing and $h(x)$ defined as in (1)–(4), then a robust algorithm will be considered for the problem (P).

3. Nonisolated optimal solution

An isolated optimal solution even if computable is often difficult to implement practically because of its instability under small perturbations of the constraints. Therefore, for solving problem (P) we only consider nonisolated feasible solutions of (P) from a practical point of view, which should be of interest. This motivates the following definitions.

Definition 3.1. A feasible solution x of (P) is called a nonisolated feasible solution if

$$B(x, \varepsilon) \cap (X \backslash \{x\}) \neq \emptyset,$$

where $B(x, \varepsilon)$ is a ball with radius any $\varepsilon > 0$ and center x; X is a feasible set of the problem (P).

Definition 3.2. A nonisolated feasible solution x^* of (P) is called a nonisolated optimal solution if $g(x^*) \leqslant g(x)$ for all nonisolated feasible solutions x of (P), i.e. if

$$g(x^*) = \min\{g(x) \mid x \in X_0^*\},$$

where X_0^* denotes the set of all nonisolated feasible solutions of (P).

Definition 3.3. Assume

$$\{x \in X_0 \mid h(x) > 0\} \neq \emptyset. \tag{5}$$