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# A modified gradient projection algorithm for solving the elastic demand traffic assignment problem

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ABSTRACT

This paper develops a path-based traffic assignment algorithm for solving the elastic demand traffic assignment problem (EDTAP). A modified path-based gradient projection (GP) method combined with a column generation is suggested for solving the equivalent excess-demand reformulation of the problem in which the elastic demand problem is reformulated as a fixed demand problem through an appropriate modification of network representation. Numerical results using a set of real transportation networks are provided to demonstrate the efficiency of the modified GP algorithm for solving the excess-demand formulation of the EDTAP. In addition, a sensitivity analysis is conducted to examine the effects of the scaling parameter used in the elastic demand function on the estimated total demand, number of generated paths, number of used paths, and computational efforts of the modified GP algorithm.



## 1. Introduction

An elastic-demand traffic assignment problem (EDTAP) is one that establishes the equilibrium between supply functions and demand functions in a transportation network. In the traffic assignment problem, the supply functions are determined by the link travel time characteristics on the network, and the demand functions are determined by the user benefits derived from travel [15]. At equilibrium, the link flows, link travel times, path flows, path travel times, origin–destination (O–D) travel demands, and O–D travel times are consistent and satisfy the user equilibrium (UE) conditions [30,28]. That is, the travel times on all used paths between any O–D pair are equal, and are also equal to or less than the travel times on any unused paths. In addition, the O–D travel demands should satisfy the demand functions. Beckmann et al. [3] provided the first convex programming formulation for the user equilibrium (UE) traffic assignment problem with endogenously determined travel demands. Based on this seminal work, many researchers have considered different formulation approaches and solution algorithms to enhance the modeling realism and applications of the UE model with elastic demand. In terms of formulation approaches, Aashtiani [1] gave the first nonlinear complementarity problem (NCP) formulation for modeling the interactions in a multimodal network. Dafermos [13] offered a variational inequality (VI) formulation for the multimodal traffic equilibrium model with elastic demand, where the link travel costs depend on the entire link flow vector and the travel demands depend on the entire mode-specific O–D cost vector. Fisk and Boyce [14] provided alternative VI formulations for the network equilibrium travel choice problem, which does not required invertibility of the travel demand function. Cantarella [7] provided a fixed point (FP) formulation for the multi-mode multi-user equilibrium assignment with elastic demand, where users have different behavioral characteristics as well as different choice sets.

As for solution algorithms, the convex combinations method (or the Frank–Wolfe algorithm) used for solving the UE traffic assignment problem with fixed demand is perhaps the most commonly used approach as it can be readily adapted to solve the elastic demand version with minor additional computational effort needed to compare the current shortest path cost with the current value of the inverse demand function [28]. Gartner [17,18] summarized three approaches for modeling the generalized traffic equilibrium problem as an equivalent network in which the elastic demand functions are represented by appropriate generating links: (1) minimum-cost circulation, (2) zero-cost overflow, and (3) excess demand. Fukushima [16] explored the possibility of solving the EDTAP via its dual problem as a nonsmooth convex programming formulation, while Nagurney [25] extended the concept of equilibration operator introduced by Dafermos and Sparrow [12] for solving the excess demand reformulation of the EDTAP. Babonneau and Vial [2] proposed a variant of the analytic center cutting plane method for solving the EDTAP with emphasis on

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large-scale instances and high accuracy. Aashtiani [1] developed one of the early path-based linearization algorithms for solving the multimodal traffic assignment problem formulated as a NCP. Cantarella [7] developed two convergent algorithms based on flow and cost averaging for solving the FP formulation. For a review of the traffic assignment algorithms, readers can refer to Patriksson [26,27].

The gradient projection (GP) algorithm has been shown as a successful path-based algorithm for solving the UE traffic assignment problem with fixed demand [22,8]. Under an ingenious approach that utilizes the special structure of the traffic assignment problem, GP only needs to perform a simple projection on the nonnegative orthant in each iteration; therefore, the required computational effort is modest. In addition, GP adopts the diagonal inverse Hessian approximation as a scaling matrix and uses the *one-at-a-time* flow update strategy to equilibrate path flows one O–D pair at a time. These features make the GP algorithm computationally more efficient than the disaggregate simplicial decomposition (DSD) algorithm (see Chen et al. [8]) for a detailed computational study between these two path-based algorithms). Thanks to its good algorithmic features, the path-based GP algorithm has been recently adapted to solve the non-additive traffic equilibrium problem [11] and the C-logit stochastic user equilibrium problem [31,32]. Both problems are for the fixed demand case, and have reported promising results. The purpose of this paper is to adapt the path-based GP algorithm for solving the EDTAP, and demonstrate that the modified GP algorithm can be as efficient as the original GP algorithm for solving the UE traffic assignment problem with fixed demand.

The paper is organized as follows. The EDTAP is briefly reviewed in Section 2. The path-based GP algorithm, its modifications for solving the excess demand formulation of EDTAP, and a simple illustration of the modified GP algorithm are discussed in Section 3. In Section 4, numerical experiments are conducted to examine the efficiency and robustness of the modified GP algorithm. Finally, some conclusions are summarized in Section 5.

## 2. Elastic demand traffic assignment problem

In this section we review the elastic demand traffic assignment problem (EDTAP) formulated as a convex program and its excess demand reformulation.

### 2.1. Convex programming formulation

As aforementioned, the EDTAP was originally proposed by Beckmann et al. [3]. In this section, we review the convex programming formulation for the EDTAP formulated in the path-based domain [28]:

$$\text{Minimize } Z_{ED} = \sum_{a \in A} \int_0^{x_a = \sum_{rs \in RS} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ka}^{rs}} t_a(w) dw - \sum_{rs \in RS} \int_0^{q_{rs}} D_{rs}^{-1}(w)\, dw \tag{1}$$

$$\text{subject to : } \sum_{k \in K_{rs}} f_k^{rs} = q_{rs}, \quad rs \in RS \tag{2}$$

$$f_k^{rs} \geq 0, \quad rs \in RS,\ k \in K_{rs} \tag{3}$$

$$q_{rs} \geq 0, \quad rs \in RS \tag{4}$$

where $A$ is the set of links; RS is the set of O–D pairs; $K_{rs}$ is the set of routes between O–D pair $rs$; $x_a$ is the flow on link $a$; $t_a(w)$ is the travel time on link $a$; $f_k^{rs}$ is the flow on path $k$ between O–D pair $rs$; $\delta_{ka}^{rs}$ is equal to 1 if link $a$ is on path $k$ between O–D pair $rs$, and 0 otherwise; $q_{rs}$ is the demand between O–D pair $rs$; and $D_{rs}^{-1}(w)$ is the inverse demand function between O–D pair $rs$, which is equal to the minimum travel time of O–D pair $rs$ at equilibrium.

The objective function in Eq. (1) consists of two terms: an UE term (i.e., the well-known Beckmann transformation) reflecting the *congestion* effect and an inverse demand term reflecting the *elasticity* of O–D demands in terms of the network level of service (LOS). Eq. (2) ensures that the path flows add up to the travel demand between each O–D pair. Non-negativity constraints on the path flows and O–D flows are imposed in Eqs. (3) and (4) to ensure meaningful solutions.

### 2.2. Excess demand formulation

The EDTAP accounts for both trip generation (i.e., travel choice) and traffic assignment (i.e., route choice) simultaneously by considering the equilibration between supply and demand. At equilibrium, the travel demand determined by the elastic demand function is consistent with the network level of service via the minimum O–D travel time for all O–D pairs. Thus, the EDTAP may be viewed as a simple combined travel demand model that considers trip generation, trip distribution, and trip assignment [26]. As suggested by Gartner [17,18], the EDTAP can be reformulated as a fixed demand problem using an equivalent network in which the elastic demand functions are represented by appropriate generating links: (1) minimum-cost circulation, (2) zero-cost overflow, and (3) excess demand (see Sheffi [28] for details of these three approaches). In this study, we adopt the excess demand formulation. From Eq. (1), the second term $\sum_{rs \in RS} \int_0^{q_{rs}} D_{rs}^{-1}(w)\, dw$ can be express as

$$\sum_{rs \in RS} \int_0^{q_{rs}} D_{rs}^{-1}(w)\, dw = \sum_{rs \in RS} \int_0^{\bar{q}_{rs}} D_{rs}^{-1}(w)\, dw - \sum_{rs \in RS} \int_{q_{rs}}^{\bar{q}_{rs}} D_{rs}^{-1}(w)\, dw, \tag{5}$$

where $\bar{q}_{rs}$ is the upper bound demand between O–D pair $rs$.

The first term $(\sum_{rs \in RS} \int_0^{\bar{q}_{rs}} D_{rs}^{-1}(w)\, dw)$ on the right-hand-side of Eq. (5) is a constant; thus, it can be dropped from the objective function since it would not affect the optimization problem. The second term $(-\sum_{rs \in RS} \int_{q_{rs}}^{\bar{q}_{rs}} D_{rs}^{-1}(w)\, dw)$ represents the excess demand. With the excess demand variable $(e_{rs} = \bar{q}_{rs} - q_{rs})$, the second term can be re-defined as follows:

$$-\sum_{rs \in RS} \int_{q_{rs}}^{\bar{q}_{rs}} D_{rs}^{-1}(w)\, dw = -\sum_{rs \in RS} \int_{\bar{q}_{rs} - q_{rs}} D_{rs}^{-1}(\bar{q}_{rs} - v)(-dv) = -\sum_{rs \in RS} \int_0^{e_{rs}} W_{rs}(v)\, dv, \tag{6}$$

where $e_{rs}$ is the excess demand variable between O–D pair $rs$, and $W_{rs}(\cdot)$ is the excess demand function between O–D pair $rs$. Fig. 1 provides a graphical illustration of the change of variable from the original demand $(q_{rs})$ to excess demand $(e_{rs})$ and the corresponding elastic demand function $(D_{rs}(\cdot))$ and excess demand function $(W_{rs}(\cdot))$.

Using the excess demand variable, the EDTAP formulation given in Eqs. (1)–(4) can be reformulated as follows:

$$\text{Minimize } Z_{ED} = \sum_{a \in A} \int_0^{x_a = \sum_{rs \in RS} \sum_{k \in K_{rs}} f_k^{rs} \delta_{ka}^{rs}} t_a(w)\, dw + \sum_{rs \in RS} \int_0^{e_{rs}} W_{rs}(v)\, dv \tag{7}$$

$$\text{subject to : } \sum_{k \in K_{rs}} f_k^{rs} + e_{rs} = \bar{q}_{rs}, \quad rs \in RS \tag{8}$$

$$f_k^{rs} \geq 0, \quad rs \in RS,\ k \in K_{rs} \tag{9}$$

$$e_{rs} \geq 0, \quad rs \in RS \tag{10}$$

where $e_{rs}$ is the excess demand variable between O–D pair $rs$, $W_{rs}(\cdot)$ is the excess demand function between O–D pair $rs$, and $\bar{q}_{rs}$ is the upper bound demand between O–D pair $rs$.