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On exact solutions for the Minmax Regret Spanning Tree problem

Francisco Pérez-Galarce [a,1], Eduardo Álvarez-Miranda [b,c,1], Alfredo Candia-Véjar [b,*], Paolo Toth [c]

[a] Programa de Magíster en Gestión de Operaciones, Universidad de Talca, Merced 437, Curicó, Chile
[b] Departamento de Modelación y Gestión Industrial, Universidad de Talca, Merced 437, Curicó, Chile
[c] DEI, Università di Bologna, Bologna, Italy

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ABSTRACT

The Minmax Regret Spanning Tree problem is studied in this paper. This is a generalization of the well-known Minimum Spanning Tree problem, which considers uncertainty in the cost function. Particularly, it is assumed that the cost parameter associated with each edge is an interval whose lower and upper limits are known, and the Minmax Regret is the optimization criterion. The Minmax Regret Spanning Tree problem is an NP-Hard optimization problem for which exact and heuristic approaches have been proposed. Several exact algorithms are proposed and computationally compared with the most effective approaches of the literature. It is shown that a proposed branch-and-cut approach outperforms the previous approaches when considering several classes of instances from the literature.

1. Introduction

The classical (deterministic) Minimum Spanning Tree (MST) problem is a fundamental problem in combinatorial optimization, and it can be applied in several areas like logistics or telecommunications. It consists of finding a spanning tree of minimum total cost in a connected and undirected graph with non-negative edge costs. Very simple and fast greedy algorithms are able to solve large MST instances in a few seconds. See [1] for algorithms and applications of the MST.

The purpose of this paper is to present exact approaches for the Minmax Regret Spanning Tree (MMR-ST) problem, a generalization of the MST, where the problem is to find a feasible solution that is ϵ-optimal for any possible realization of the vector of the objective function parameters, with ϵ as small as possible. The objective function parameters are the costs of the edges of the graph and each of them is associated with a real cost interval. It is supposed that there is independence among the different cost intervals and that the uncertainty is only considered in the cost function. Problems with this type of data uncertainty are known as Interval Data Minmax Regret problems; for other types of uncertain data (see [2,6]).

It is known that many MMR combinatorial optimization problems are NP-Hard even if the corresponding deterministic version is polynomially solvable; for example, the Shortest Path problem and the Assignment problem are NP-Hard in their MMR versions (MMR-P and MMR-A, respectively). Only for few problems, the corresponding MMR counterpart is polynomially solvable [see [6]]. Several exact and heuristic approaches have been proposed for different MMR problems including MMR-ST [22,17,16,19,10,12], MMR-P [9,15,10], MMR-A [20], MMR Set Covering [21], MMR-TSP [18].

Literature review: It is known that the MMR-ST is also an NP-Hard problem [5,4]; therefore, the existing exact algorithms are able to solve only small instances. In [22], a compact formulation is presented and a set of instances (`Ya`) comprised by up to 25 nodes are solved by using CPLEX.

Later on, in [3], a constraint programming algorithm for the MMR-ST was developed; this method outperformed the one proposed in [22], allowing to solve to optimality instances of a new class (`He1`) with up to 40 nodes. In [17], a branch-and-bound algorithm was designed and applied to the `Ya` instances and to a new group of complete graph instances (`Mo`). For both classes of instances, the proposed algorithm outperformed the exact approach developed in [3].

A Benders Decomposition (BD) algorithm for the MMR-ST was proposed in [16], and it was used to solve `Ya` and `Mo` instances. For the first group of instances, the BD approach solved all the instances to optimality, outperforming the results reported by [22,3,17]. For the second set of instances, the author considered a parameter p to control the width of the cost intervals; this allowed to conclude that the performance of the algorithm depended strongly on the value of p (the larger the p was, the more difficult the optimization task became).

With respect to the heuristic approaches for the MMR-ST, three classes of algorithms are found in the literature: (i) the two “one-scenario” heuristics HM and HU, the first proposed in [11] (where it is shown that it has an approximation ratio 2) and the second

* Corresponding author. Tel.: +56 75 201706.
E-mail addresses: acandia@utalca.cl, candia.vejar@gmail.com (A. Candia-Véjar).
[1] The authors wish it to be known that, in their opinion, the first two authors should be regarded as joint First Authors.

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proposed in [18]; (ii) a simulated annealing (SA) proposed in [19]; and (iii) a tabu search (KMZ-TS) proposed in [12]. The SA approach was applied to small instances of the MMR-ST (up to 30 nodes) and reasonable results were obtained; the author pointed out that the approach should also work properly for large instances due to the search scheme used in the algorithm. In [12], the KMZ-TS algorithm was extensively tested on different sets of instances (`Ya`, `He1`, `Mo` and others), and it is shown that its performance is remarkably better than the one reported for the SA.

Our contribution and paper outline: Different algorithmic strategies for solving the MMR-ST to optimality are proposed. More precisely, a BD and a branch-and-cut approach are designed to solve benchmark instances that extend the size of instances for which an exact algorithm gets optimal solutions or small gaps. Additionally, the obtained lower bounds allow us to improve the knowledge about the quality of the solutions given by the approach proposed in [12].

In Section 2 basic notation and special results for the MMR-ST are presented. Section 3 presents two mathematical programming formulations which will be used later. The proposed algorithms are described in detail in Section 4. Computational results and their analysis are presented in Section 5. Conclusions and future work are presented in Section 6.

2. Minmax Regret Spanning Tree (MMR-ST)

Let $G=(V,E)$, where $|V|=n$ and $|E|=m$, be an undirected connected graph with V being the set of nodes and E being the set of edges. Suppose that for every edge $e\in E$ an interval $[c_e^-,c_e^+]$ is given $(0\le c_e^-\le c_e^+)$. The values c_e^+ and c_e^- will be referred as the upper and lower limit, respectively, of the corresponding interval. It is assumed that the cost of edge $e\in E$ can take any value on its corresponding interval, independent of the values taken by the cost of other edges. Let Γ be defined as $\Gamma=\otimes_{e\in E}[c_e^-,c_e^+]$, i.e., the set of all possible realizations of edge costs. Thus, an element $s\in\Gamma$ is a so-called *scenario*, because it represents a particular realization of edge costs; these costs will be denoted by c_e^s for each $e\in E$. Let $\mathbf{X}\in\{0,1\}^{|E|}$ be a binary vector such that $X_e=1$ if $e\in E$ belongs to a spanning tree of G and $X_e=0$ otherwise. For a given scenario s and a given vector $\mathbf{X}$, the cost of the corresponding spanning tree is given by $F_s(\mathbf{X})=\sum_{e\in E(\mathbf{X})}c_e^s$, where $E(\mathbf{X})$ corresponds to the subset of edges such that $X_e=1$, $\forall e\in E(\mathbf{X})$ and $X_e=0$ otherwise. The classical MST for a fixed scenario $s\in\Gamma$ is

$$F_s^*=\min\{F_s(\mathbf{X})|\mathbf{X}\in\Phi\}, \qquad \text{(MST)}$$

where Φ is the set of all binary vectors associated with spanning trees of G.

For a fixed $\mathbf{X}\in\Phi$ and $s\in\Gamma$, the function $R(s,\mathbf{X})=F_s(\mathbf{X})-F_s^*$ is called the *regret* for $\mathbf{X}$ under scenario s. For a given $\mathbf{X}\in\Phi$, the *worst-case regret* or *robust deviation* is defined as

$$Z(\mathbf{X})=\max\{R(s,\mathbf{X})|s\in\Gamma\}. \qquad \text{(MR)}$$

The minmax regret version of the MST problem (MMR-ST) is given by the following:

$$Z^*=\min\{Z(\mathbf{X})|\mathbf{X}\in\Phi\}. \qquad \text{(MMR)}$$

In [22], it is shown that an optimal solution for the right-hand-side of (MR) (the worst-case scenario for a given $\mathbf{X}$) holds the following property.

Theorem 1 (*Yaman et al. [22]*). *The worst-case scenario for a solution X, $s^{\mathbf{X}}$, is obtained when the cost of the edges in $E(\mathbf{X})$ is set to the corresponding upper limits and the cost of all other edges to the corresponding lower limits, i.e., $c_e^{s^{\mathbf{X}}}=c_e^+$, $\forall e\in E(\mathbf{X})$ and $c_e^{s^{\mathbf{X}}}=c_e^-$, $\forall e\in E\backslash E(\mathbf{X})$.*

Combining the previous property with (MMR), one can derive the following formulation for the MMR-ST.

$$Z_{MMR}^*=\min\sum_{e\in E(\mathbf{X})}c_e^+-\theta \qquad (1)$$

$$\text{s.t.}\quad \theta\le\sum_{e\in E(\mathbf{Y})}c_e^-+\sum_{e\in E(\mathbf{Y})\cap E(\mathbf{X})}(c_e^+-c_e^-),\quad \forall\mathbf{Y}\in\Phi \qquad (2)$$

$$\theta\in\mathbb{R}_{\ge 0}\quad\text{and}\quad \mathbf{X}\in\Phi. \qquad (3)$$

Note that this formulation has an exponential number of constraints (2) (one per each spanning tree of G).

Let s^M be the scenario defined by $c_e^{s^M}=1/2(c_e^-+c_e^+)$, $\forall e\in E$. An important algorithmic result for a wide class of MMR problems (including MMR-ST) was provided by [11] using s^M, where an approximation algorithm of ratio 2 was designed; the result reads as follows:

Lemma 1 (*Kasperski and Zieliński [11]*). *Let $\mathbf{X}^M$ be a minimum spanning tree for the midpoint scenario s^M. This solution holds $Z(\mathbf{X}^M)\le 2Z_{MMR}^*$.*

Thus, a solution with an approximation ratio at most 2, X^M, is obtained by simply solving a classical MST problem on G with edge costs defined by s^M. In practice, these approximate solutions have shown a good performance [see, e.g., [18,10]].

The solution obtained for the scenario s^+ defined by the upper limits of the intervals, i.e., $c_e^{s^+}=c_e^+$, has also shown an interesting performance [see, e.g., [18,10,12]], although it has been proved that this solution can be arbitrarily bad [see [10]]. Both solutions, X^M and X^+, will be used as part of the exact approaches proposed in this work.

3. MIP formulations for the MMR-ST

Notation: Let $r\in V$ be an arbitrary node of V which we will denote as the *root* node. Let A be the set of arcs of the bi-directed counterpart of G, $G_A=(V,A)$, such that $A=\{(i,j),(j,i)|e:\{i,j\}\in E\}$; likewise, $c_{ij}^-=c_{ji}^-=c_e^-$ and $c_{ij}^+=c_{ji}^+=c_e^+$ $\forall e:\{i,j\}\in E$.

3.1. Formulation#1

This first formulation is based on directed cut-set inequalities. The Linear Programming relaxation of this type of formulations usually provides good quality lower bounds, since many facet-inducing inequalities can be projected out of the directed model for optimal tree problems [8]. Consequently, instead of looking for a spanning tree of G we look for a spanning arborescence of G_A.

Let $\mathbf{x}\in\{0,1\}^{|A|}$ be a binary vector such that $x_{ij}=1$ if arc $(i,j)\in A$ belongs to a spanning arborescence of G_A and $x_{ij}=0$ otherwise. We will use the following notation: a set of nodes $S\subseteq V$ $(S\neq\emptyset)$ and its complement $\overline{S}=V\backslash S$, induce two directed cuts: $\delta^+(S)=\{(i,j)|i\in S,j\in\overline{S}\}$ and $\delta^-(S)=\{(i,j)|i\in\overline{S},j\in S\}$. A vector $\mathbf{x}$ is associated with a directed spanning tree of G_A (spanning arborescence) rooted at r if it satisfies the following set of inequalities:

$$\sum_{(i,j)\in\delta^-(S)}x_{ij}\ge 1,\quad \forall S\subseteq V\backslash\{r\}\ S\neq\emptyset \qquad (4)$$

$$\sum_{(i,j)\in\delta^-(j)}x_{ij}=1,\quad \forall j\in V\backslash\{r\}. \qquad (5)$$

Constraints (4), which are exponential in number, are known as *cut-set* or *connectivity* inequalities and they ensure that there is a directed path from the root r to each node $v\in V\backslash\{r\}$. This type of constraints is usually used in the context of effective branch-and-cut procedures (see, e.g., [13]). Its separation can be performed in polynomial time using a maximum-flow algorithm on a *support graph* with arc-capacities given by the current fractional solution