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New relaxation-based algorithms for the optimal solution of the continuous and discrete p-center problems

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ABSTRACT

We present new relaxation algorithms for the uncapacitated continuous and discrete p-center problems. We have conducted an experimental study that demonstrated that these new algorithms are extremely efficient.

1. Introduction

The p-center problem (see, for example, [1]), also known as the minimax location–allocation problem, deals with the optimal location of emergency facilities. The locations of n demand points are given, and we need to locate p service facilities. The *value* of a candidate solution to the p-center problem is the maximum distance between a demand point to its nearest service facility. Our objective is to find the solution with the minimal value; we want to locate the service facilities so as to minimize the maximum distance between a demand point to its nearest service facility. It is assumed that all the facilities perform the same kind of service, and that the number of demand points that can get service from a given center is unlimited.

Relaxation (in the context of this paper) [1,2] is a method to *optimally* solve a large location problem by solving a succession of small sub-problems. It is an iterative algorithm which updates, at each step, bounds on the optimal solution, until the optimal solution is reached. This paper presents new relaxation algorithms for the p-center problem.

Every step of a relaxation algorithm involves solving a p-center-like problem on a subset of the demand points. Our input is the subset and a value r, which is called the *coverage distance*. We need to answer: "Is there a solution to the sub-problem with value less than r?".

The new relaxation algorithms we describe try to reduce the number of iterations, or reduce the sizes of sub-problems, or reduce the values of the coverage distances (or a combination of these factors), so that we can improve performance, and therefore solve larger problems to optimality.

There are two main variants of the p-center problem in the literature; they differ by the possible location of the service points. Many authors deal with the *continuous* problem in which the points to be located optimally can be anywhere in the plane, but another interesting problem is the *discrete* case where there is a finite set of potential points (x_j, y_j) out of which one wishes to find the points which fulfill the minimax condition. In some cases, weights w_i are associated with the service points (a_i, b_i). Another classification of the problems is associated with the relevant metrics. In many cases, the distances between demand and service points are Euclidean (e.g., [3]). Also considered are problems where the distances are defined by minimal distances on a graph; this variant was first solved by Minieka [4].

The formulation of the Euclidean unweighted p-center problem is

$$\min_{X_1,\ldots,X_p} \left\{ \max_{1\leqslant i\leqslant n} \left[\min_{1\leqslant j\leqslant p} r_{ij} \right] \right\}$$

where $X_j = (x_j, y_j)$ for $j = 1, \ldots, p$ is the location of the new facility and $r_{ij} = [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}$. Megiddo and Supowit [5] have shown that both the p-center and p-median problems are NP-hard and that it is NP-hard even to approximate the p-center problems sufficiently closely. On the other hand, Hochbaum [6] has shown that given certain assumptions on the input distribution, there are polynomial algorithms that deliver a solution asymptotically close to the optimum with probability that is asymptotically one.

Most of the methods developed for solving the continuous Euclidean problem are geometrical in nature. When we are looking for a single service point ($p = 1$), the solution of the problem will be the center of the smallest circle enclosing n given points in the plane (see e.g., [7]). This can occur in one of two ways. The smallest circle

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can be determined by three demand points on its circumference or, alternatively, by two points on the two ends of a diameter. In the former case, the three points are the edges of an acute triangle [8]. The geometrical methods are based on a sophisticated search for the smallest enclosing circle among the circles built on subsets of two and three demand points. This includes the repeated solution of relaxed, smaller sub-problems as described below in the broader context of the p-center problem.

Chen [9] suggested a method that enables both the solution of the minisum and minimax location–allocation continuous problems by using a differentiable approximation to the objective function and solving it by using nonlinear programming. This enabled the solution of relatively large problems, but the result was not necessarily optimal since local minima may have been reached. Drezner [3,10] presented heuristic and optimal algorithms for the p-center problem in the plane. The heuristic method yielded results for problems with up to $n=2000$ and $p=10$ whereas the optimal method solved problems with up to $n=30$, $p=5$ or $n=40$, $p=4$. Watson-Gandy [11] suggested an algorithm that can optimally solve problems with up to about 50 demand points and 3 centers in reasonable time. The p-center problem on networks has been solved by Minieka [4] and by Toregas et al. [12]. A finite method, which is rather inefficient for large problems was suggested. An improvement based on the use of relaxations was offered by Handler and Mirchandani [1]. In Section 2 we elaborate on relaxation methods.

Some other papers dealing with the continuous p-center problem include the following. Hwang et al. [13] describe a slab-dividing approach, which is expected to efficiently solve the Euclidean p-center problem. These authors show that their algorithm has time complexity of $O(n^{O(\sqrt{p})})$. Suzuki and Drezner [14] propose heuristic procedures and upper bounds on the optimal solution where the demand points are distributed on a square. One of the methods they use employs the Voronoi heuristic. The same method has been recently used by Wei et al. [15]; the authors explore the complexity of solving the continuous space p-center problem in location planning. Agarwal and Sharir [16] discuss efficient approximate algorithms for problems in geometric optimization, which include the Euclidean p-center in d dimensions. Hale and Moberg [17] give a broad review on location problems, which includes the Euclidean p-center problem.

The discrete p-center problem is also known to be NP-hard [18]. For a review on discrete network location models see Current et al. [19]. Daskin [20] presents an optimal algorithm which solves the discrete p-center problem by performing a binary search over possible solution values. This algorithm solves maximal covering sub-problems, rather than the set-covering sub-problems solved by Minieka [4]. Mladenović et al. [21] present a basic Variable Neighborhood Search and two Tabu Search heuristics for the p-center problem without the triangle equality. Elloumi et al. [22] present a new integer linear programming formulation for the discrete p-center problem and show how to use this new formulation to obtain tight bounds on the optimal solution. They use these bounds in an exact solution method and report very good computational results. Recent works on the discrete problem include algorithms given by Caruso et al. [23] and by Ilhan et al. [24]. The latter authors describe an efficient exact method for the discrete p-center problem. A tight lower bound to the optimal value is found in an initial phase of the algorithm, which consists of solving linear programming sub-problems. Good computational results are reported for each of an extensive list of test problems derived from OR-Lib and TSP-Lib problems with up to 900 data points.

We present new relaxation algorithms. We report excellent computational results for both the continuous and discrete cases. We solved problems taken from OR-Lib [25] and TSP-Lib [26].

The rest of the paper is structured as follows. Section 2 explains the principles of relaxation and present new relaxation algorithms. In Section 3 we present the results of our experimental study. Section 4 contains conclusions and open problems.

2. Relaxation algorithms for the p-center problem

2.1. The p-center problem

The p-center problem, also known as the minimax location–allocation problem, deals with the optimal location of emergency facilities. We are given the locations of n demand points, and the objective is to locate p service facilities so as to minimize the maximum distance between a demand point to its nearest service facility.

Here is an equivalent way of looking at the same problem: we are given the locations of n demand points. We need to locate p circles that will cover all of the demand points. Our objective is to minimize the radius of the maximal circle. Clearly, this is exactly the p-center problem, where the centers of the p circles are the locations of the p service facilities.

When we consider the second interpretation of the p-center problem, we say that a set of p circles is a *feasible* solution to the problem, if the circles cover all of the demand points. When we consider the first interpretation, then any set of p points is considered a *feasible* solution to the p-center problem. Whenever convenient, we will alternate between the two equivalent definitions of the p-center problem.

2.2. Theory

Relaxation is a simple method to *optimally* solve a large location problem by solving a succession of small sub-problems. Although one cannot know in advance how many sub-problems need to be solved, once the global optimum is reached, it is identified as such. This as opposed to some heuristic methods which usually yield local minima. Though in the worst case relaxation may be very slow, it is usually very efficient.

Chen and Handler [2] adapted the relaxation method, previously suggested for the solution of location problems on networks [1], to the problem in continuous Euclidean two-dimensional space. In the solution of the p-center problem there is usually only one circle which is critical in the sense that two or three demand points are on its circumference. There is much freedom in the exact position of the other circles and therefore, in the location of all but one of the centers. The value of the solution is determined by the radius of this critical circle, whereas the radii of the other circles may vary in size below this critical value. Thus, the number of possible optimal solutions is usually infinite. Chen and Handler [2] proved a theorem stating that among all the optimal solutions to the minimax problem of serving n demand points in Euclidean space by p service points, there is at least one in which all demand points are covered by critical circles, the largest of which has a radius r_p, which is the value of the solution. With the aid of this theorem, the search can be reduced to a finite number of critical circles.

The number of critical circles to be considered is $\binom{n}{3}+\binom{n}{2}+n$, where $\binom{n}{3}$ is the number of circles determined by three points on their circumference, $\binom{n}{2}$ is the number of circles defined by two points determining the diameter and n is the number of null circles; a null circle is a service point located at a demand point, the former serving only the latter. The number of possible combinations to cover n points by p critical circles becomes very large when n is large. However, geometrical considerations associated with a known upper bound and with the properties of relevant triangles defined by demand points, significantly reduce the size of the sub-problem to be solved.