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An interior point cutting plane heuristic for mixed integer programming

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ABSTRACT

We explore the use of interior point methods in finding feasible solutions to mixed integer programming. As integer solutions are typically in the interior, we use the analytic center cutting plane method to search for integer feasible points within the interior of the feasible set. The algorithm searches along two line segments that connect the weighted analytic center and two extreme points of the linear programming relaxation. Candidate points are rounded and tested for feasibility. Cuts aimed to improve the objective function and restore feasibility are then added to displace the weighted analytic center until a feasible integer solution is found. The algorithm is composed of three phases. In the first, points along the two line segments are rounded gradually to find integer feasible solutions. Then in an attempt to improve the quality of the solutions, the cut related to the bound constraint is updated and a new weighted analytic center is found. Upon failing to find a feasible integer solution, a second phase is started where cuts related to the violated feasibility constraints are added. As a last resort, the algorithm solves a minimum distance problem in a third phase. The heuristic is tested on a set of problems from MIPLIB and CORAL. The algorithm finds good quality feasible solutions in the first two phases and never requires the third phase.

1. Introduction

Besides its practical importance, finding feasible solutions for mixed integer programming (MIP) is an important step towards finding optimal solutions. Good feasible solutions help fathom branches earlier in the branch and bound tree and contribute to the reduction of the computational time and memory required.

Compared to extreme points, the motivation behind the use of interior points resides in the fact that its rounding is more likely to result in a feasible integer solution. Choosing a central point such as the center of gravity would be ideal. Its calculation, however, is hard. The analytic center is easier to calculate and its location can be displaced by duplicating certain constraints, i.e. modifying their weights. In this paper, weights are used to guide the analytic center towards regions where rounding will likely give a feasible integer solution.

Some of the early attempts to use interior point methods for integer programming are due to Mitchell and Todd [1] and Mitchell [2] who use a primal-dual predictor-corrector interior point method in a cutting plane method to solve integer programs. Being aware that warm starting is key to a successful method, Mitchell [2] uses an idea from Gondzio [3] where the interior point method is terminated early at a central interior point that is later used to warm start the solution methodology after cuts are added. Afterwards, the concept of analytic centers was mainly and successfully used in cutting plane methods [4–6]. Its use in integer programming was to solve master problems in Lagrangian relaxation/column generation settings [7]. This paper describes a novel application of analytic centers to integer feasibility problems and introduces the notion of integer analytic centers.

The paper makes a valuable contribution towards the use of interior point methods in mixed integer programming. We show that through the use of the analytic center cutting plane method (ACCPM), interior point methods can compete with linear programming based methods in finding quality feasible solutions for MIP. Due to the nature of interior point methods and our implementation, the computational times, although within a maximum of 3 min, are not as competitive. The paper also introduces the notion of an integer analytic center and the way to compute it, which renders the application of the analytic center cutting plane method possible to problems where the master problem is an integer problem such as in Benders decomposition [8].

The literature is rich with heuristics for MIP. Hillier [9] was among the first to propose a heuristic based on interior points. His three-phase method starts by identifying an interior path connecting an interior point and the optimal solution of the LP relaxation. In Phase-II, a search around the interior path is conducted to find a feasible integer solution, but with no guarantees for a successful termination. In Phase-III, the algorithm attempts to find other feasible integer solutions that improve the objective function value. Hillier's algorithm was implemented within a branch and bounds algorithm in Jeroslow and Smith [10]. Balas and Martin [11] use the fact that every 0–1 binary problem is equivalent to a linear problem with all slack variables being basic. Their proposed heuristic solves the linear programming relaxation

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and pivots all the slack variables to the basis. The extension of [11] to general mixed integer problems is presented in Balas et al. [12]. Saltzman and Hillier [13] describe an algorithm that enumerates feasible 1-ceiling points, integer solutions lying on or near the boundary of the feasible region, around the optimal linear programming solution. Glover and Laguna [14,15] propose a heuristic framework based on cut search procedures for general mixed integer problems. Løkketangen and Glover [16] present a tabu search heuristic while Balas et al. [17] propose an algorithm that enumerates extended facets of the octahedron to solve 0–1 binary problems. The feasibility pump for 0–1 binary problems was introduced in Fischetti et al. [18] and was extended to general mixed integer problems in Bertacco et al. [19]. In Achterberg and Berthold [20], a modification of the feasibility pump that improves the quality of the feasible solutions is presented.

The proposed approach, that we refer to as analytic center feasibility method (ACFM), is composed of three phases. In the first, a search is conducted around two line segments connecting the analytic center and two extreme points of the LP relaxation. The candidate points are first rounded and checked for feasibility. The cut that is related to the objective function is updated as feasible solutions are identified, and a new analytic center is found before restarting the search. Upon failing to find a feasible integer solution, a second phase is started where the weights of the violated constraints are incremented and a cut formed from the convex combination of the violated constraints is added. If no feasible solution is found, a third phase is invoked in which a minimum distance problem is solved guaranteeing a successful termination of ACFM.

ACFM is tested on a set of problems from MIPLIB and CORAL. The algorithm found good quality feasible solutions in less than 20 iterations of the first two phases, without the need for the third phase. In 32 of the 36 tested problems, ACFM found a better quality solution than the feasibility pump, at the expense of taking more computational time. Furthermore, ACFM found a feasible solution for four instances in which the feasibility pump has failed.

The rest of the paper is organized as follows. Section 2 presents the main results of the paper related to the analytic center feasibility method. Computational results are presented in Section 3. Section 4 summarizes the paper and highlights future research.

2. The analytic center feasibility method

This section is devoted to the description of the analytic center feasibility method for mixed integer programming. We first focus on the analytic center, introduce the notion of integer analytic center, and then use it in an iterative algorithm to find feasible solutions for MIP.

2.1. The weighted analytic center

The analytic center was first introduced in [21] and used in [4] in a cutting plane algorithm. To provide a detailed description, let us introduce the generic mixed integer problem:

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \le c, \\ & y_j \text{ integer } \forall j \in J \end{aligned} \tag{1}$$

that is of interest in this work. We assume that $A^T y \le c$ has the implicit or explicit bound constraints $l \le y \le u$. The analytic center associated with the LP relaxation

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \le c \end{aligned}$$

is defined as the point that maximizes the product of the distances from the boundary of the localization set:

$$\mathcal{F} = \left\{ \begin{array}{ll} y: & b^T y \le z_u \\ & A^T y \le c \end{array} \right\},$$

where z_u is an upper bound on $b^T y$ that ensures that the objective function value of the analytic center is within a desired bound. The weighted analytic center adds a weight on a particular constraint to push the analytic center away from it. Goffin and Vial [22] show that repeating a constraint is equivalent to setting a weight on its corresponding slack in the potential function. Usually, a weight equal to the number of constraints is associated with the bound constraint $b^T y \le z_u$ to force the analytic center away from the upper bound. Given a weight ν_i corresponding to each constraint i, the weighted analytic center y^{ac} is the unique point that maximizes the weighted potential function:

$$\begin{aligned} \max \quad & \varphi_D = \sum_{i=0}^{m} \nu_i \ln s_i \\ \text{s.t.} \quad & A^T y + s = c, \\ & b^T y + s_0 = z_u, \\ & s_0,\ s > 0. \end{aligned} \tag{2}$$

Taking the dual of problem (2) we get:

$$\begin{aligned} \max \quad & \varphi_P = -c^T x - z_u x_0 + \sum_{i=0}^{m} \nu_i \ln x_i \\ \text{s.t.} \quad & Ax + bx_0 = 0, \\ & x_0,\ x > 0. \end{aligned} \tag{3}$$

The necessary and sufficient first order optimality conditions of (2) and (3) are

$$Sx = \nu,$$

$$s_0 x_0 = \nu_0,$$

$$Ax + bx_0 = 0, \quad x, x_0 > 0,$$

$$A^T y + s = c, \quad s > 0,$$

$$b^T y + s_0 = z_u, \quad s_0 > 0.$$

Defining $\tilde{S}$ as the diagonal matrix of $\tilde{s} = \left[\begin{smallmatrix} s_0 \\ s \end{smallmatrix}\right]$, $\tilde{N}$ as the diagonal matrix of $\tilde{\nu} = \left[\begin{smallmatrix} \nu_0 \\ \nu \end{smallmatrix}\right]$, $\tilde{x} = \left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right]$, $\tilde{c} = \left[\begin{smallmatrix} z_u \\ c \end{smallmatrix}\right]$, and $\tilde{A} = [b, A]$, the first order optimality conditions are rewritten as

$$\tilde{S}\tilde{x} = \tilde{\nu},$$

$$\begin{aligned} & A\tilde{x} = 0, \quad \tilde{x} > 0, \\ & \tilde{A}^T y + \tilde{s} = \tilde{c}, \quad \tilde{s} > 0. \end{aligned} \tag{4}$$

In this paper, we use the Newton method to solve problem (3). Starting from a strictly feasible point x, the algorithm calculates the Newton direction d_x:

$$d_x = N^{-(1/2)} X (\nu^{1/2} - N^{-(1/2)} X s),$$

$$s = c - A^T y,$$

$$y = (AN^{-1} X^2 A^T)^{-1} AN^{-1} X^2 c,$$

and updates x as:

$$x^+ = x + \alpha d_x.$$

A measure of proximity to the analytic center is defined as $\eta_P(x) = \| \nu^{1/2} - N^{-(1/2)} X s \|$. When far from the analytic center $\eta_P(x) > 1$, the step size α is found by performing a line search along d_x to maximize φ_P subject to $x^+ > 0$. When near the analytic center $\eta_P(x) < 1$, a full