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Formulations and exact algorithms for the vehicle routing problem with time windows

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Abstract

In this paper we review the exact algorithms proposed in the last three decades for the solution of the vehicle routing problem with time windows (VRPTW). The exact algorithms for the VRPTW are in many aspects inherited from work on the traveling salesman problem (TSP). In recognition of this fact this paper is structured relative to four seminal papers concerning the formulation and exact solution of the TSP, i.e. the arc formulation, the arc-node formulation, the spanning tree formulation, and the path formulation. We give a detailed analysis of the formulations of the VRPTW and a review of the literature related to the different formulations. There are two main lines of development in relation to the exact algorithms for the VRPTW. One is concerned with the general decomposition approach and the solution to certain dual problems associated with the VRPTW. Another more recent direction is concerned with the analysis of the polyhedral structure of the VRPTW. We conclude by examining possible future lines of research in the area of the VRPTW.

Keywords: Vehicle routing; Time windows; Exact algorithms

1. Introduction

In 1959, a paper by Dantzig and Ramser [1] appeared in the journal Management Science concerning the routing of a fleet of gasoline delivery trucks between a bulk terminal and a number of service stations supplied by the terminal. The distance between any two locations is given and a demand for a certain product is specified for the service stations. The problem is to assign service stations to trucks such that all station demands are satisfied and total mileage covered by the fleet is minimized. The authors imposed the additional conditions that each service station is visited by exactly one truck and that the total demand of the stations supplied by a certain truck does not exceed the capacity of the truck. The problem formulated in the paper by Dantzig and Ramser [1] was given the name 'truck dispatching problem'. I do not know who coined the name 'vehicle routing problem' (VRP) for Dantzig and Ramser's problem but it caught on in the literature and is the title of the most recent book on the problem, and some of its main variants, edited by Toth and Vigo [2]. In this book, Toth and Vigo [3] considered branch and bound algorithms for the VRP, Naddef and Rinaldi [4] branch and cut algorithms for the VRP and polyhedral studies, Simchi-Levi [5] set covering based approaches for the VRP, Cordeau et al. [6] the VRP with time windows, Toth and Vigo [7] the VRP with backhauls, and Desaulniers et al. [8] the VRP with pickup and delivery. Furthermore, the book reviews heuristic approaches and issues arising in real-world applications. Now the basic variant of the VRP is often given the name 'capacitated vehicle routing problem'

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(CVRP) to distinguish it from other members of the family of VRPs. In this paper we consider the VRP with time windows (VRPTW), where each customer must be visited within a specified time interval, called a time window. We consider the case of hard time windows where a vehicle must wait if it arrives before the customer is ready for service and it is not allowed to arrive late. In the case of soft time windows a violation of the time window constraints is accepted but then a price must be paid.

Dantzig and Ramser [1] described how the VRP may be considered as a generalization of the traveling salesman problem (TSP). They described the generalization of the TSP with multiple salesmen and called this problem the 'clover leaf problem', a name that is the very picture of the problem. If there are m salesmen we will refer to the clover leaf problem as the m-TSP, a less lucid name. If in the m-TSP we impose the condition that specified deliveries be made at every location, excepting the start location, we get Dantzig and Ramser's problem. Obviously the VRP is identical with the m-TSP if the total demand of all locations is less than the capacity of a single vehicle. The standard reference book on the TSP was edited by Lawler et al. [9]. In this book Hoffman and Wolfe [10] describe how the importance of the TSP comes from the fact that it is typical of other problems of its genre: combinatorial optimization.

Dantzig had previously collaborated with Fulkerson and Johnson in developing an exact algorithm to the TSP. The appearance of their paper 'Solution of a large-scale traveling-salesman problem' [11] in the journal Operations Research was according to Hoffman and Wolfe [10] "one of the principal events in the history of combinatorial optimization". In this paper the authors first associated with every tour a vector whose entries are indexed by the roads between the cities. An entry of this vector is 1 whenever the road between a pair of cities is traveled, otherwise it is 0. They also defined the linear equations that ensure all cities are visited exactly once in all representations of tours. These equations are called the degree constraints. Second, they defined a linear objective function that expressed the cost of a tour as the sum of road distances of successive pairs of cities in the tour. The problem is then to minimize the linear objective function such that the degree constraints are satisfied and the solution forms a tour. Third, the authors made a linear programming problem out of this integer programming problem by identifying just enough additional linear constraints on the vectors to assure that the minimum is assumed by some tour. This lead to the introduction of the subtour elimination constraints, which excludes solutions where cities are visited exactly once, but in a set of disconnected subtours. However, the authors pointed out that there are other types of constraints which sometimes must be added in addition to subtour elimination constraints in order to exclude solutions vectors involving fractional entries.

By now the approach of Dantzig et al. [11] is basic in combinatorial optimization. The approach is concerned with identifying linear inequalities or cutting planes describing the polytope defined by the convex hull of the points in the Euclidean space that represents the set of feasible solutions of the combinatorial optimization problem. No full description of the TSP polytope is known and because the TSP belongs to the class of NP-complete combinatorial optimization problems there is no hope for a polynomial-time cutting plane method for the TSP, unless NP=P. However, as Dantzig et al. [11] showed the cutting plane algorithm can still be applied to the TSP by including the TSP polytope in a larger polytope (a relaxation) over which we minimize a linear objective function. In this way the TSP is formulated as a linear program that gives a lower bound for the TSP which can be useful in a branch and bound algorithm. Padberg and Rinaldi [12] refined the integration of the enumeration approach of classical branch and bound algorithms with the polyhedral approach of cutting planes to create the solution technique called branch and cut. This method has been very successful in solving large-scale instances of the TSP and different authors have therefore applied the polyhedral approach to other hard combinatorial optimization problems. Laporte et al. [13] were the first to apply the polyhedral approach to the VRP. Finally, we note that the field of discrete mathematics where combinatorial optimization problems are formulated as linear programs is called polyhedral combinatorics and we refer to the recent work of Schrijver [14] for a detailed treatment of this subject. For a treatment of polyhedral theory we refer to Nemhauser and Wolsey [15].

Now we consider another basic method in combinatorial optimization which is concerned with the characterization of the objective function of the combinatorial optimization problem instead of its polytope. Using relaxation and duality we can determine the optimal objective function value, or at least a good lower bound on it (assuming minimization), without explicitly solving the integer problem. In particular, we are concerned with Lagrangian relaxation and duality. A related technique is Dantzig–Wolfe decomposition, which provides an equivalent bound to the Lagrangian dual bound. In Lagrangian relaxation a set of complicating constraints are dualized into the objective function by associating Lagrangian multipliers with them. This gives us an infinite family of relaxations with respect to the Lagrangian multipliers. For a given set of values of the Lagrangian multipliers the relaxed problem is called the Lagrangian subproblem. The problem of determining the largest lower bound for this family is called the Lagrangian dual problem. A fundamental