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A spectral method for bonds☆

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Abstract

We present an spectral numerical method for the numerical valuation of bonds with embedded options. We use a CIR model for the short-term interest rate. The method is based on a Galerkin formulation of the partial differential equation for the value of the bond, discretized by means of orthogonal Laguerre polynomials. The method is shown to be very efficient, with a high precision for the type of problems treated here and is easy to use with more general models with nonconstant coefficients. As a consequence, it can be a possible alternative to other approaches employed in practice, specially when a calibration of the parameters of the model is needed to match the observed market data.

Keywords: Bonds; Embedded options; Spectral methods; Laguerre polynomials

1. Introduction

A bond is a contract paying a known fixed amount, the principal, at a given date in the future, called the maturity. The contract can pay smaller quantities, the coupons, periodically at specific dates up to the maturity date. When it gives no coupons it is called a zero-coupon bond. Many contracts incorporate one or several embedded options such as call, put or conversion features. A call option gives the issuer the right to purchase back the bond for a fixed quantity called the call price of the bond. A put option gives the holder the right to return the bond to the issuer for a specified amount known as the put price of the bond. Convertible bonds add the possibility of exchanging the bond for a fixed number of shares of the issuer. As it has been pointed out in [1], the embedded options are an integral part of the contract and cannot be traded independently of the bond.

The optional features of the contract, call, put or conversion option, can usually be exercised at given dates during a specified period of the life of the bond, that is, the contract allows for early exercise so that the bond with its embedded options are an American-style interest rate derivative. As it is well-known, approximate methods, fully numerical or semi-analytical, are necessary for pricing American style contracts, as there are no known closed-form formulae even in the simpler cases. A wide variety of numerical methods have been used in the literature for the numerical valuation of bonds with embedded options. Finite-difference methods were used in [2], whereas trinomial trees were later used in [3]. Finite elements coupled with the characteristic method for two factor convertible bonds were used in [4,5] where the author used the method of lines and an efficient implicit–explicit Runge–Kutta method for the time integration.

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In [6], a semi-analytical approach based on the Green function coupled with numerical integration were employed for the valuation of callable bonds with notice period. The same problem was solved in [7] by means of a finite volume method with stabilization and in [8] by means of a finite element/characteristic method. Recently, in [1] the authors propose a dynamic programing approach which is proved to be a very efficient way to numerically evaluate callable bonds with notice period.

In this paper, we present a spectral numerical method for the valuation of such bonds. The numerical procedure we present here is based in a Galerkin formulation of the relevant model-dependent partial differential equation for the value of the bond, discretized by means of orthogonal Laguerre polynomials. The method is proved to be very efficient; it shows a high precision for the type of problems we treat here and it is easy to use with general models with nonconstant coefficients. Although, for brevity, we shall use a CIR model [9] for the dynamics of the short-term interest rate, the methodology can be easily extended to other type of models, such as Vasicek [10] or Hull and White [3] models, for example. It is worth remarking that time-dependent models are needed in practice to calibrate the parameters of the model to match the observed yield curves, see [4]. Calibration of parameters is a difficult problem, specially for American derivatives. Mathematically, it is a inverse problem for a nonlinear equation, see [11]. This kind of problem is usually very sensitive to the computed values of the solution of the equation and therefore extremely precise numerical results are needed to correctly solve it. The method we present here can be a possible alternative to other approaches employed in practice such as finite differences or finite elements.

A common problem shared by all the numerical methods mentioned above is the necessity of introducing artificial boundaries in order to discretize the infinite interval in which the state variable, i.e. the interest rate, is a priori defined. Although it is possible to show convergence for the approximations defined in increasing finite intervals, see, for example, [12,13], in practice this is the cause of difficulty in controlling numerical errors which, if not treated appropriately, can ruin any attempt to calibrate the parameters of the model to match the observed market data. In this paper, we present a numerical method based on Laguerre polynomials. The method does not require any artificial boundary and it can be extremely efficient at least for a class of problems related to the valuation of bonds.

2. A model problem

Several diffusion models have been used to describe the dynamics of the short-term risk-free interest rate. In this paper, we consider the Cox, Ingersoll and Ross (CIR) model [9] for the short-term interest rate described by the stochastic differential equation

$$\mathrm{d}r_t = k(\bar{r} - r_t)\,\mathrm{d}t + \sigma r_t^{1/2}\,\mathrm{d}B_t, \quad 0 \leqslant t \leqslant T, \tag{1}$$

where $\{B_t, t \geqslant 0\}$ is a standard Brownian motion, k is the reverting rate, $\bar{r}$ is the reverting level and σ is the volatility. The numerical method we present is easily adaptable to other alternatives models as Vasicek model [10] or models with time-dependent parameters such as Hull and White model [3].

Let $P(r, t, T)$, $0 \leqslant t \leqslant T$, be the value at time t of a zero coupon bond with maturity date T and principal scaled to 1. By standard hedging arguments

$$P(r, t, T) = \mathbb{E}^Q \left[\exp\left(-\int_t^T r(s)\,\mathrm{d}s \right) \middle| r(t) = r \right], \tag{2}$$

where $\mathbb{E}^Q[\cdot | r(t) = r]$ represent the expectation conditional on $r(t) = r$ with respect to the risk-neutral probability measure Q. Equivalently, $P(r, t, T)$ represents the discount factor over the period $[t, T]$ when $r(t) = r$. A partial differential equation for the price of the bond under the CIR model is

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} + (k(\bar{r} - r) - qr)\frac{\partial P}{\partial r} - rP = 0, \tag{3}$$

where qr is the risk premium corresponding to the choice of $q(r, t) = -qr^{1/2}/\sigma$ as market price of risk.

Eq. (3) together with the final condition $P(r, T, T) = 1$ is a well-posed problem. Note that the partial differential equation (3) is singular at $r = 0$. No boundary condition at $r = 0$ is needed as long as the condition $k\bar{r} \geqslant \sigma^2/2$ is satisfied. We refer to [7] for a study of the problem of assigning boundary conditions at the singular boundary.