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Post-optimality analysis of the optimal solution of a degenerate linear program using a pivoting algorithm

Jan Stallaert

University of Connecticut, Storrs, CT, USA

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Abstract

This paper gives a theory and method that specifies how the optimal solution of a linear program changes when the right-hand side of the original problem is changed and when the original optimal solution exhibits primal degeneracy. The method determines an optimal change vector as the resource availabilities change, and it calculates a range for which this vector is valid. Resource availabilities are allowed to change simultaneously in any arbitrary proportion, and the concept of an "efficient resource bundle" is introduced. The geometry of the optimal change vector is presented from which the desired results are derived. The connection between the geometrical results and their algebraic calculation in tableau-form is shown. Our method uses a pivoting algorithm and the relationship with post-optimality results from interior-point methods will be established.

Keywords: Linear programming; Post-optimality analysis; Shadow prices; Simplex method; Degeneracy

1. Introduction

Linear progamming (LP) textbooks often end their discussion of the simplex method by referring to a post-optimality analysis that can easily be performed by means of the final simplex tableau. However, it is then mentioned that caution has to be taken when either primal or dual solution is degenerate. In that case, usual post-optimality results do no longer hold. For example, when the primal solution is degenerate, it is

E-mail address: stallaert@sba.uconn.edu.

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possible that the change in the optimal solution when the right-hand side changes, is only valid in a zero range, which is rather useless from a practical perspective. Another unique phenomenon that occurs for degenerate problems is the complementarity effect between resources. For example, if in the degenerate case resources a and b have shadow prices of $p(a) \geqslant 0$ and $p(b) \geqslant 0$, respectively, then it is possible that acquiring a and b in a certain proportion, say in the amounts r_a of a and r_b of b yields for the value of the bundle: $p_r(a, b) > r_a p(a) + r_b p(b)$, so the value of the bundle $p_r(a, b)$ is worth more than the weighted sum of the values of resources a and b separately (for non-degenerate problems equality always holds). At first glance, a complimentarity effect seems rather surprising for a problem that consists of all linear functionals. Jansen et al. [1] describe the problems with commercial LP packages and outline three approaches how to deal with the difficulties caused by degeneracy.

Research on post-optimality analysis using simplex tableaus goes back more than 20 years [2–4], but the most recent stream uses interior point methods to obtain post-optimality results in the presence of non-uniqueness and degeneracy. Given an optimal interior point solution, an optimal partition can be identified [5] which can then be used for sensitivity analysis in the presence of degeneracy. Adler and Monteiro [6] find all breakpoints of the parametric objective function when the perturbation vector r is kept constant. To compute every breakpoint, an Oracle LP problem with an auxiliary column needs to be solved. Yildirim and Todd [7] describe the computation of the optimal solution to a perturbed system in one interior point iteration. As in the latter paper, this paper primarily focuses on *the optimal change vector*, i.e., the direction in which the optimal *solution* changes as the resource availabilities are varied, but we use an extreme point method to arrive at those conclusions. If the optimal solution and the optimal change vector are both unique, the interior and extreme point methods of course give the same optimal change vector.

Many papers on post-optimality analysis (e.g., [1–4,8–11] focus on how the *optimal value* of the LP changes when the right-hand side changes, and construct methods on how to find the breakpoints and the rate of change (shadow price) of a piecewise linear *optimal value* function. The purpose there is to find the shadow prices u_r of some resource (combination) r and the breakpoints τ such that the optimal value takes the expression $v_r^*(t) = wy_r^*(t) = wy^* + tu_r \; \forall t \in (0, \tau]$, where wy^* is the value of the original optimal solution. Our focus of attention, however, is on how the change in certain resource availabilities affects the optimal solution, for the following reasons. The primary driver of this research is the appearance of market-based decomposition-like algorithms [12,13] where different iterations require the solution of perturbed problems where resource availabilities are varied and where the knowledge of the optimal change vector is needed to compute ask and bid prices. In addition, the proof of finite convergence of such algorithms hinges upon the generation of *extreme points* of the perturbed system at each step, hence our use of extreme point methods rather than interior point methods. Another reason for our interest in the *optimal change vector* rather than shadow prices of the resource(s) is practical. When LP is used to solve planning problems (e.g., production planning, etc.) managers may sometimes be more interested in knowing how the optimal solution changes as a function of a right-hand side change, because many times data for the latter have been forecast and is subject to change before the plan is executed. For example, managers might be more interested in knowing how their production *plan* would change when demand forecasts for the different products change (or when resource availabilities display some variance) rather than the effect of such change on the objective function, in order to anticipate the change in production plan and take the necessary precautions (e.g., not take a machine down for maintenance if it is likely to become a bottleneck when forecasts change).

In the rest of the paper, when we mention “degeneracy”, we mean “primal degeneracy”. That is, we do not study the effects of dual degeneracy and multiple primal optima. In case we have multiple primal