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Decision Support

On solving matrix games with pay-offs of triangular fuzzy numbers: Certain observations and generalizations

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ABSTRACT

The purpose of this paper is to highlight a serious omission in the recent work of Li (2012) for solving the two person zero-sum matrix games with pay-offs of triangular fuzzy numbers (TFNs) and propose a new methodology for solving such games. Li (2012) proposed a method which always assures that the max player gain-floor and min player loss-ceiling have a common TFN value. The present paper exhibits a flaw in this claim of Li (2012). The flaw arises on account of Li (2012) not explaining the meaning of solution of game under consideration. The present paper attempts to provide certain appropriate modifications in Li's model to take care of this serious omission. These modifications in conjunction with the results of Clemente, Fernandez, and Puerto (2011) lead to an algorithm to solve matrix games with pay-offs of general piecewise linear fuzzy numbers.

1. Introduction

Li (2012) in his recent paper proposed a new method to solve two person zero-sum matrix games with pay-offs of TFN's. He emphasized, and in fact illustrated, that the method proposed by him always assures a common TFN-type fuzzy value for both Player I gain-floor and Player II loss-ceiling functions. Therefore, he concluded that any matrix game with pay-offs of TFNs has a TFN-type fuzzy value. He further emphasized that the conclusion (*i.e.*, a common TFN value for both players) is rational because the underlying fuzzy matrix game is a zero-sum game. He also argued that his proposed method results in superior performance as compared to the existing methods in literature, more specifically, Campos (1989), Bector, Chandra, and Vidyottama (2004), Li (1999); Li and Yang (2004) and Li (2008). This is because the existing methods either do not provide a TFN value for the game or even if they do so the value is not common for the two players.

In this paper, we demonstrate that although Li's first conclusion that the value of the game is a TFN is correct but his other conclusion that both players have a common TFN value is flawed. The mistake in Li's work is observed due to omission of an essential concept of 'solution of a game'. In this paper, we proceed with a thorough investigation of Li's work and highlight a serious omission in an effort to alert future adoption of Li's approach in fuzzy matrix games. We also suggest appropriate modifications to take care of this omission. These modifications in conjunction with the results of Clemente, Fernandez, and Puerto (2011) lead to an algorithm to solve matrix games with pay-offs of general piecewise linear fuzzy numbers.

The remainder of the paper is organized as follows. Section 2 reviews and points out a serious omission in the method of Li (2012). Section 3 provides our revision in order to resolve the mistakes in Li's research. A simple numerical example is presented in Section 4. Section 5 describes an algorithm to solve matrix games with pay-offs of general piecewise linear fuzzy numbers. This algorithm is based on our revision of Li's work as suggested in Section 3. Some concluding remarks are furnished in Section 6.

2. Review of Li's method

Rather than presenting the mathematical details of Li's method to solve a two person zero-sum matrix game with pay-offs of TFNs we consider only the numerical example presented in his work. This has been done to keep the presentation short and also to clearly illustrate how and why we differ from his point of view. The example is cited from Campos (1989) and it has now become almost a bench mark example in the area of fuzzy matrix games (see, Bector & Chandra, 2005). Let the fuzzy matrix game be described by the

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fuzzy matrix $\widetilde{A}$,

$$\widetilde{A} = \begin{pmatrix} (175, 180, 190) & (150, 156, 158) \\ (80, 90, 100) & (175, 180, 190) \end{pmatrix},$$

where each entry is a TFN, and notation (a_l, a_m, a_r) has the usual interpretation of TFN.

Li (2012) associated six linear programming problems with game representation $\widetilde{A}$, three for Player I and three for Player II. These are numbered (31), (33) and (35) for Player I, and numbered (23), (24) and (27) for Player II in his work (Li, 2012). The optimal values for three linear programming problems for Player I are given by $V^m = V^R(1) = 161.0526$, $V^r = V^R(0) = 166.3934$, $V^l = V^L(0) = 155.2083$, respectively. The other three linear programming problems for Player II are solved to get optimal values $W^m = W^R(1) = V^R(1)$, $W^r = W^R(0) = V^R(0)$ and $W^l = W^L(0) = V^L(0)$. Thereby it is concluded that the fuzzy value of the matrix game $\widetilde{A}$ is given by the common TFN (155.2083, 161.0526, 166.3934).

Going through the work of Li (2012) what we observed that he mainly had used α-cuts, interval-valued fuzzy matrix game theory (Collins & Hu, 2008; Li, 2011a, 2011b; Li & Cheng, 2002), and the deterministic linear programming duality, to draw his conclusions.

What Li (2012) has talked about is the value of the game only but it is important to note that a solution of a game does not mean only its value but it also very much includes the optimal strategies for the two players. Though Li (2012) takes care of one component of solution of a fuzzy matrix game (*i.e.* value), he completely neglects the second aspect of a solution and that is optimal strategies of two players in a game. In fact a more serious omission is that Li (2012) never defined what he meant by a 'solution of a fuzzy matrix game' and therefore missed about the optimal mixed strategies altogether. Here lies our first point of concern in Li's work. Even if one agrees, for the time being, that (V^l, V^m, V^r) is the common TFN value of the game, the major question remains about optimal strategies. Li is obviously silent about them, primarily because the three values V^l, V^m and V^r describing TFN as value of the game were obtained for three different optimal vectors $y = (y_1, y_2)^T$, namely $y^{R_*}(1)$, $y^{R_*}(0)$ and $y^{L_*}(0)$, respectively. In fact there is absolutely no way to describe the optimal strategies for two players following the approach of Li (2012).

Besides missing on the solution concept, we also disagree with his other argument which he strongly emphasized that since the fuzzy matrix game is a zero-sum game we must have a common value for two players as the same holds in crisp zero-sum matrix games. But we have no reasons to expect this assertion since the problem set-up is fuzzy and 'equality' of two TFNs representing optimal values $\widetilde{V}$ for Player I and $\widetilde{W}$ for Player II cannot be taken in the usual crisp sense rather it has to be understood in an altogether different sense of fuzzy. We shall be discussing this and the other related aspects in the section to follow.

Our contention here is that though the value of the game for each player indeed will be TFN, $\widetilde{V} = (V^l, V^m, V^r)$ for Player I and $\widetilde{W} = (W^l, W^m, W^r)$ for Player II, but these will neither be equal nor optimal for the overall game as suggested in Li (2012). Consequently there is no question of a common TFN being the optimal value of the game.

3. Our revision on Li (2012) model

With regard to fuzzy matrix games and related topics we shall follow the notations and terminologies of Bector and Chandra (2005). We hope there will be no confusion even if they are somewhat different from Li (2012).

Let $\mathbf{R^n}$ denote the n-dimensional Euclidean space and $\mathbf{R^n_+}$ be its non-negative orthant. Let $e^T = (1, \ldots, 1)$ be a vector of 'ones' whose dimension is specified as per the specific context. By a two person zero-sum fuzzy matrix game FG, we mean the triplet FG $= (S^p, S^n, \widetilde{A})$ where $S^p = \{x \in \mathbf{R^p_+},\ e^T x = 1\}$, $S^n = \{y \in \mathbf{R^n_+},\ e^T y = 1\}$ and $\widetilde{A} = (\widetilde{a}_{ij})_{p\times n}$ with $\widetilde{a}_{ij}$ $(i = 1, \ldots, p,\ j = 1, \ldots, n)$ is a fuzzy number. In the terminology of fuzzy matrix game theory, S^p and S^n are called the strategy spaces for Player I and Player II, respectively, and $\widetilde{A}$ is called the fuzzy pay-off matrix of FG. If all fuzzy numbers $\widetilde{a}_{ij}$ are TFNs then FG is called the fuzzy matrix game with pay-offs of triangular fuzzy numbers. Our discussion here is concerned with such games only as these are the one considered by Li (2012). Thus, in this paper, FG shall always mean $(S^p, S^n, \widetilde{A})$ with all entries $\widetilde{a}_{ij}$ in $\widetilde{A}$ being TFNs.

The TFN $\widetilde{a}$ will be denoted as per the standard notation $\widetilde{a} = (a_l, a_m, a_r)$. Also if we are given two TFN's $\widetilde{a} = (a_l, a_m, a_r)$ and $\widetilde{b} = (b_l, b_m, b_r)$, then we write $\widetilde{a} \preceq \widetilde{b}$ if $a_l \leq b_l, a_m \leq b_m$ and $a_r \leq b_r$. The symbol $\widetilde{a} \succeq \widetilde{b}$ is understood analogously. Further $\widetilde{a} \prec \widetilde{b}$ if $\widetilde{a} \preceq \widetilde{b}$ with at least one of the three inequalities $a_l \leq b_l, a_m \leq b_m$ and $a_r \leq b_r$ holding as strict inequality. Here we must note that this ordering '$\preceq$' in TFN's is a partial order. This is important in understanding the meaning of solution of game FG, because it tells that the maximality (or minimality) of fuzzy numbers on a set has to be understood in terms of partial order '$\preceq$' only. Therefore it makes sense to denote the matrix game FG as FG $= (S^p, S^n, \widetilde{A}, \preceq)$ with $\widetilde{a}_{ij} = ((a_{ij})_l,\ (a_{ij})_m, (a_{ij})_r)$.

We have the following definitions (see, Bector & Chandra, 2005).

Definition 3.1. (Reasonable solution of FG) Let $\widetilde{v} = (v_l, v_m, v_r)$ and $\widetilde{w} = (w_l, w_m, w_r)$ be TFN's. Then $(\widetilde{v}, \widetilde{w})$ is called a reasonable solution of the game FG if there exist $\bar{x} \in S^p$, and $\bar{y} \in S^n$ such that

(i) $\bar{x}^T \widetilde{A} y \succeq \widetilde{v}$ for all $y \in S^n$, and
(ii) $x^T \widetilde{A} \bar{y} \preceq \widetilde{w}$ for all $x \in S^p$.

If $(\widetilde{v}, \widetilde{w})$ is a reasonable solution of FG then $\widetilde{v}$ and $\widetilde{w}$ are respectively called the *reasonable value for Player I* and the *reasonable value for Player II*.

Definition 3.2. (Solution of FG) Let D and E denote the set of all reasonable values $\widetilde{V}$ and $\widetilde{W}$ for Player I and Player II, respectively. An element $(\widetilde{v}^*, \widetilde{w}^*) \in D \times E$ is called a *solution of the game* FG if $\nexists$ $(\widehat{v}, \widehat{w}) \in D \times E$ such that

(i) $\widehat{v} \succ \widetilde{v}$ for all $\widetilde{v} \in D$, and
(ii) $\widehat{w} \prec \widetilde{w}$ for all $\widetilde{w} \in E$.

Let $x^* \in S^m$ and $y^* \in S^n$ be strategies for which $(\widetilde{v}^*, \widetilde{w}^*)$ is a solution of game FG as per Definition 3.2. Then $(x^*, y^*, \widetilde{v}^*, \widetilde{w}^*)$ is called a complete solution of the game FG. However in the sequel we shall write for $(x^*, y^*, \widetilde{v}^*, \widetilde{w}^*)$ a solution only. Thus in our discussion to follow, a solution of the game FG will consists of the value pair $(\widetilde{v}^*, \widetilde{w}^*)$ along with the strategy pair (x^*, y^*).

If $(x^*, y^*, \widetilde{v}^*, \widetilde{w}^*)$ is a solution of FG, then x^* (respectively y^*) is called an optimal strategy for Player I (respectively Player II) and $\widetilde{v}^*$ (respectively $\widetilde{w}^*$) is called the value of FG for Player I (respectively Player II).

In view of above definitions, solving the game FG is equivalent to solving the following two multiobjective linear programming problems.

$$\begin{array}{lll} (\text{MOP} - \text{I}) & \text{Max} & (v_l, v_m, v_r) \\ & \text{subject to} & \\ & & \displaystyle\sum_{i=1}^{p} (a_{ij})_l x_i \geq v_l, \quad (j = 1, \ldots, n), \\ & & \displaystyle\sum_{i=1}^{p} (a_{ij})_m x_i \geq v_m, \quad (j = 1, \ldots, n), \\ & & \displaystyle\sum_{i=1}^{p} (a_{ij})_r x_i \geq v_r, \quad (j = 1, \ldots, n), \\ & & e^T x = 1, \quad x \geq 0, \end{array}$$