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A mixed-type Galerkin variational formulation and fast algorithms for variable-coefficient fractional diffusion equations

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We consider the variable-coefficient fractional diffusion equations with two-sided fractional derivative. By introducing an intermediate variable, we propose a mixed-type Galerkin variational formulation and prove the existence and uniqueness of the variational solution over $H_0^1(\Omega) \times H^{\frac{1-\beta}{2}}(\Omega)$. On the basis of the formulation, we develop a mixed-type finite element procedure on commonly used finite element spaces and derive the solvability of the finite element solution and the error bounds for the unknown and the intermediate variable. For the Toeplitz-like linear system generated by discretization, we design a fast conjugate gradient normal residual method to reduce the storage from $O(N^2)$ to $O(N)$ and the computing cost from $O(N^3)$ to $O(N \log N)$. Numerical experiments are included to verify our theoretical findings. Copyright © 2017 John Wiley & Sons, Ltd.

Keywords: mixed-type Galerkin variational formulation; fractional diffusion equation; finite element; solvability; error bounds; fast algorithm

1. Introduction

In this paper, we consider the following fractional-order diffusion equation of order $2-\beta$ with $0<\beta<1$:

$$
\begin{aligned}
&-\left(\theta\, {}_0D_x^{1-\beta} + (1-\theta)\, {}_xD_1^{1-\beta}\right)(K(x)Dc) = f, \\
&x \in \Omega = (0,1), c(0) = c(1) = 0.
\end{aligned} \tag{1.1}
$$

Problem (1.1) arises in the mathematical modeling of transport processes that exhibit anomalous conservative diffusion [1–6], in which the unknown c stands for the concentration of a solute, $K(x)$ is the diffusivity coefficient with

$$0 < K_{\min} \le K(x) \le K_{\max} < +\infty\ ,\ x \in (0,1),$$

$f \in L^2(\Omega)$ is the source or sink term, and $0 \le \theta \le 1$ indicates the relative weight of forward versus backward transition probability. We let D be the first-order differential operator, ${}_0D_x^{1-\beta}$ and ${}_xD_1^{1-\beta}$ refer to, respectively, the left and right Riemann–Liouville fractional derivative operators of order $1-\beta$ defined by (2.2) and (2.4).

Because the analytic methods, such as Fourier, Laplace, and Mellin transforms, are available only in a few limited cases for (1.1), our attention turns to numerical methods. In the last decade, the finite difference method, the finite volume method, the spectral method, the multi-grid method, and the fast difference method have been developed for space-fractional partial differential equation (see [7–13] and the references therein).

Unlike second-order diffusion equation, the structure of the solution for fractional diffusion equation has not been clarified so far. It is meaningful to formulate a kind of Galerkin formulation to induce a easily computed finite element procedure as well as predict the

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behaviors of the true solution, such as the solvability and regularity. In this line, Ervin and Roop [14–16] developed a Galerkin formulation and finite element method for the fractional elliptic differential equations with a constant diffusivity coefficient and established the $H^{1-\frac{\beta}{2}}(\Omega)$−regularity of the solution. Since then, the discontinuous Galerkin method [17], the Petrov–Galerkin method [18–20], and the expanded mixed finite element method [21] appeared for fractional diffusion equation with constant coefficient. Wang and Yang [19] pointed out that the idea in [14–16] can not be extended directly to the variable-coefficient case because the coercivity of bilinear form may not be ensured. As a remedy, they proposed a discontinuous Petrov–Galerkin formulation (DPG) [20] for variable-coefficient fractional diffusion equations with one-sided fractional derivative. However, the DPG needs to carefully choose appropriate finite-dimensional trial space and test space to ensure the weak coercivity.

In this paper, we propose a mixed-type Galerkin variational formulation by introducing the flux function $u = -K(x)Dc$ as an intermediate variable and prove the existence and uniqueness of the variational solution over the space $H_0^1(\Omega) \times H^{\frac{1-\beta}{2}}(\Omega)$ for variable-coefficient FDE (1.1) with two-sided fractional derivatives. Under the assumption that the solution has certain regularity, we establish the equivalence between (1.1) and the variational formulation. We develop the mixed-type finite element procedure on commonly used finite element spaces, then derive the solvability of the finite element solution and the error bounds for the unknown and the intermediate variable on $H_0^1(\Omega) \times H^{\frac{1-\beta}{2}}(\Omega)$, following from the good properties of the mixed-type Galerkin variational formulation. Because the numerical method proposed in this article generates full coefficient matrix, we design a fast conjugate gradient normal residual (FCGNR) method to reduce the storage from $O(N^2)$ to $O(N)$ and the computing cost from $O(N^3)$ to $O(N \log N)$ by exploring Toeplitz-like structure of the resulting coefficient matrix.

The rest of the paper is organized as follows. In Section 2, we recall the definitions and properties of the fractional operators, the fractional derivative spaces, and its corresponding fractional Sobolev spaces. In Section 3, we introduce the flux function u as an intermediate variable to derive a mixed-type Galerkin variational formulation and prove its solvability. We also establish the equivalence between (1.1) and the variational formulation under the assumption that the solution has certain regularity. In Section 4, we design a mixed-type finite element procedure on commonly used finite element spaces, then derive the existence and uniqueness of the finite element solution and the error bounds for the unknown and the intermediate variable on $H_0^1(\Omega) \times H^{\frac{1-\beta}{2}}(\Omega)$. In Section 5, we study Toeplitz-like structure of the resulting coefficient matrix and establish an FCGNR method to reduce the storage and the computing cost. In Section 6, we perform numerical experiments to verify our theoretical results and the efficiency of the proposed method. The last section is concluding remarks.

2. Preliminaries

We first briefly review the definitions and some properties of left-sided and right-sided Riemann–Liouville fractional derivatives.

The left-sided Riemann–Liouville fractional derivative of order μ $(n-1 \le \mu < n)$ is defined by [22–24]

$$ {}_0D_x^{\mu} u = \frac{d^n}{dx^n}({}_0I_x^{n-\mu} u), \tag{2.2} $$

where ${}_0I_x^{n-\mu}$ is the fractional integral operator defined by

$$ {}_0I_x^{n-\mu} u = \frac{1}{\Gamma(n-\mu)} \int_0^x (x-s)^{n-\mu-1} u(s)ds \tag{2.3} $$

with $\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$ being the Gamma function. Similarly, the right-sided version of fractional-order derivative and integral are defined by

$$ {}_xD_1^{\mu} u = (-1)^n \frac{d^n}{dx^n}({}_xI_1^{n-\mu} u) \tag{2.4} $$

and

$$ {}_xI_1^{n-\mu} u = \frac{1}{\Gamma(n-\mu)} \int_x^1 (s-x)^{n-\mu-1} u(s)ds. \tag{2.5} $$

The fractional integral operator satisfies a semigroup property, that is, there holds for $\mu, \nu > 0$

$$ \begin{aligned} {}_0I_x^{(\nu+\mu)} u &= {}_0I_x^{\nu}\, {}_0I_x^{\mu} u, \\ {}_xI_1^{(\nu+\mu)} u &= {}_xI_1^{\nu}\, {}_xI_1^{\mu} u, \text{ for all } u \in L^2(\Omega). \end{aligned} \tag{2.6} $$

And Fubini's Theorem [25] implies that the integral by parts formula holds for $\mu > 0$,

$$ ({}_0I_x^{\mu} u, v) = (u, {}_xI_1^{\mu} v), \text{for all } u \in L^2(\Omega), \tag{2.7} $$