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Decision Support

Mean-risk analysis with enhanced behavioral content ☆

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ABSTRACT

We study a mean-risk model derived from a behavioral theory of Disappointment with multiple reference points. One distinguishing feature of the risk measure is that it is based on mutual deviations of outcomes, not deviations from a specific target. We prove necessary and sufficient conditions for strict first and second order stochastic dominance, and show that the model is, in addition, a Convex Risk Measure. The model allows for richer, and behaviorally more plausible, risk preference patterns than competing models with equal degrees of freedom, including Expected Utility (EU), Mean–Variance (M-V), Mean-Gini (M-G), and models based on non-additive probability weighting, such as Dual Theory (DT). In asset allocation, the model allows a decision-maker to abstain from diversifying in a positive expected value risky asset if its performance does not meet a certain threshold, and gradually invest beyond this threshold, which appears more acceptable than the extreme solutions provided by either EU and M-V (always diversify) or DT and M-G (always plunge). In asset trading, the model provides no-trade intervals, like DT and M-G, in some, but not all, situations. An illustrative application to portfolio selection is presented. The model can provide an improved criterion for mean-risk analysis by injecting a new level of behavioral realism and flexibility, while maintaining key normative properties.

1. Introduction

Mean-risk analysis is an appealing approach to decision under risk that has sprung abundant literature and applications. This is because measuring the value of gambles as a function of their rewards and risks goes to the heart of decision makers' concerns in a direct, transparent manner (Jia & Dyer, 1996). There seems to be general agreement—even compelling arguments (de Giorgi, 2005)—that the potential reward of a gamble should be captured by its expected value, i.e., its mean. There is less accord about what constitutes an acceptable measure of risk. The challenge is to balance desirable normative properties with intuitively or behaviorally appealing considerations. This tension ultimately lies at the heart of any prescriptive theory of choice under risk.

Here, we propose a mean-risk model that results from a reformulation of Disappointment without Prior Expectation (Delquié & Cillo, 2006), a theory of Disappointment in which every outcome of a prospect can act as a reference point for any other outcome. We show that this mean-risk model presents advantages over the standard competing models because it is able to produce solutions to mean-risk optimization problems that are behaviorally more realistic, and at the same time it retains key normative properties required for use in a wide range of applications. Due to this flexibility, our model may provide an attractive criterion to capture decision makers' risk-return preference patterns in mean-risk analysis.

The paper proceeds as follows. Section 2 introduces the model. We show how it relates to other models of risk, and that it defines a class of risk measure distinct from the classic families widely considered throughout the literature. In Section 3, we provide necessary and sufficient conditions for monotonicity with respect to first and, more importantly, second order stochastic dominance, two essential normative criteria for ordering risky prospects. This generalizes previous results concerning the Mean-Gini model (Ogryczak & Ruszczyński, 2002; Yitzhaki, 1982). In Section 4, we show that the model yields a Convex risk measure, which is highly desirable for use in risk management because it favors diversification. Next, in Section 5, we examine the model's implications for asset trading and optimal allocation. There, we show that the model allows for a richer pattern of risk taking behaviors than other standard models, and we specify the conditions under which qualitatively different types of behaviors occur. The risk taking behaviors produced by the model appear more realistic than those of other classic models with comparable degrees of freedom. These

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results are closely tied to the model's ability to bridge first order and second order risk aversion. Section 6 addresses practical issues in calibrating and using the model for applications, and provides a numerical example in stock portfolio selection. By way of summary, Section 7 concludes that the model provides a tractable, sound analysis of choice under risk, expanding the range of available solutions in mean-risk analysis. All proofs appear in the Appendices.

2. The proposed mean-risk model

The literature on risk emanates from several intellectual traditions, notably Statistics (measures of dispersion and the moments approach), Economics (the EU approach, but also the inequality measurement approach), Finance (the portfolio efficient set approach), and Psychology (the behavioral/cognitive approach). Sarin and Weber (1993) present an overview of the Risk-Value models literature at the time of their writing; Pedersen and Satchell (1998) provide a fairly detailed review of risk measures.

Expected Utility stands as *the* ultimately rational approach to choice under risk, however, there is no explicit construction of a risk index as a primitive in EU. For an individual with utility function u, the risk of a gamble X can be measured as its risk premium, defined as $\pi(X) = E[X] - u^{-1}(Eu[X])$ (Pratt, 1964). However, the valuation of a gamble, i.e., its certainty equivalent $CE(X) = u^{-1}(Eu[X])$, cannot in general be calculated directly from its expected value and its risk premium in a Risk-Value spirit, because the estimation of $\pi(X)$ usually requires calculating the certainty equivalent, leading to a circularity. Under particular conditions on the u function and/or the distribution of X, EU can take a Risk-Value form. For example, if the utility function is exponential and gambles have a normal distribution, or if the utility function is quadratic, then EU is equivalent to a Mean–Variance model. Further ways to cast EU as a function of risk and return have been explored in some depth by Bell (1995) and Jia and Dyer (1996): the possibilities seem confined to a limited set. Because the notion of risk in EU is entirely driven by the concavity of the utility function, it is completely intertwined with the concept of diminishing marginal utility of wealth. To require that the valuation of each and every risk be entirely and only determined by the pattern of utility for wealth may be too rigid for some decision makers. That is, EU may leave out some aspects of risk that legitimately matter to the decision maker.

The so-called Risk-Value framework may offer more flexibility in dealing with risk (Dyer & Jia, 1997) by allowing to define a risk measure "from scratch," that is, unconstrained by whether it is consistent with the maximization of a particular EU function. Because risk is associated with the presence of uncertainty in the payoffs, that is, the extent to which their distribution departs from a sure outcome, risk measures are germane with measures of dispersion. Risk is traditionally measured as the propensity of a random outcome to deviate from some reference level. Stone (1973) proposes that three basic ingredients are relevant to devising a risk measure: (i) a reference level, from which deviations are measured; (ii) the range of deviations taken into account; and (iii) how deviations are weighed. He shows that this defines a general family that includes the standard risk measured used in Finance: variance, semi-variance, mean absolute deviation, and the probability of a loss worse than some specified level.

Among the wide variety of risk measures that has been proposed, some have received special attention as regards their normative properties (e.g., compliance with stochastic orderings) and their computational performance in optimization. This is the case for the Mean Absolute Deviation, Semi-lower Deviation, Conditional Value-at-Risk, and the Gini Mean Difference, among others (see the work of Krzemienowski & Ogryczak, 2005; Mansini, Ogryczak, & Speranza, 2003, 2007; Ogryczak & Ruszczyński, 1999, 2002). The risk measure we propose here was motivated by a desire to account for risk preferences that deviate systematically from EU, such as the widely observed Allais' (1953) paradox and certainty effect (Kahneman & Tversky, 1979), the common ratio effect, and reference-dependence effects.

2.1. A behaviorally motivated mean-risk model

Delquié and Cillo (2006) developed the Disappointment without Prior Expectation model of choice under risk based on the postulate that individuals are liable to experience a *mixture* of disappointment and contentment from comparing the outcome received from a gamble to all the other possible outcomes, worse and better, rather than a single prior expectation. This extends the notion of reference dependence by allowing each and every outcome in the gamble to play the role of a reference point, that is, the value of an outcome is relative to the *entire* context in which it is embedded. In all previous formulations of Disappointment, including Bell (1985) and Loomes and Sugden (1986), the gamble is summarized into a single reference point. Kőszegi and Rabin (2007) proposed a model of reference-dependent risk taking behavior in which the reference level is stochastic, consisting of the expectations the decision maker held in the recent past.

It was also shown in Delquié and Cillo (2006) that Disappointment without Prior Expectation could be reformulated as a Risk-Value model, taking the following form:

$$V(X) = \sum_{i=1}^{n} p_i v(x_i) - \sum_{i=1}^{n}\sum_{j \geqslant i} p_i p_j H(v(x_i) - v(x_j)),$$

where X is a gamble that yields payoff x_i with probability p_i, $i = 1, \ldots, n$, $\sum p_i = 1$ and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$; $v(\cdot)$ is an increasing function that describes the subjective value of outcomes; and the function H describes how an individual values discrepancies between obtained and missed outcomes, that is, the loss associated with getting the lower of two outcomes. The immutable properties of H, that stem from its very definition, are: (i) $H(0) = 0$, and (ii) H is defined on the non-negative domain, that is, it takes non-negative deviations as argument, i.e., differences between ordered outcomes.

Here, for parsimony and for the sake of having a Risk-Value representation comparable to those that have appeared before, we focus on a special case of the above model: we will assume v linear throughout this paper. This assumption does not play a role in the essential results and claims developed in the paper, and it will enable us to concentrate on what can be accomplished with the simpler form. Thus, the model we are interested in here is:

$$V(X) = E[X] - \Delta(X)$$
$$\text{with } \Delta(X) = \sum_{i=1}^{n}\sum_{j \geqslant i} p_i p_j H(x_i - x_j), \tag{1}$$

where $E[X]$ is the mean of X, a measure of its potential reward, and $\Delta(X)$ defines a risk-premium, that is, the amount by which the reward will be discounted to account for the presence of risk in X. For example, for a binary gamble X with outcomes x, y with probabilities p, $1 - p$ respectively, and $x \geqslant y$, we have: $V(X) = px + (1 - p)y - p(1 - p)H(x - y)$. If the outcomes are not ordered, we can just enter their absolute difference in the H function. If F denotes the cumulative distribution of X, the continuous form of $\Delta(X)$ is:

$$\Delta(X) = \int_{-\infty}^{+\infty} \int_{-\infty}^{x} H(x - y) dF(y) dF(x)$$
$$= E\left[\int_{-\infty}^{X} H(X - y) dF(y)\right]. \tag{2}$$