European Journal of Operational Research 239 (2014) 786–793

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Investment under duality risk measure ☆

Zuo Quan Xu *

Department of Applied Mathematics, Hong Kong Polytechnic University, Kowloon, Hong Kong

ARTICLE INFO

Article history:
Received 5 August 2013
Accepted 16 June 2014
Available online 28 June 2014

Keywords:
Duality axiom
Duality risk measure
Duality index
Portfolio selection

ABSTRACT

One index satisfies the duality axiom if one agent, who is uniformly more risk-averse than another, accepts a gamble, the latter accepts any less risky gamble under the index. Aumann and Serrano (2008) show that only one index defined for so-called gambles satisfies the duality and positive homogeneity axioms. We call it a duality index. This paper extends the definition of duality index to all outcomes including all gambles, and considers a portfolio selection problem in a complete market, in which the agent's target is to minimize the index of the utility of the relative investment outcome. By linking this problem to a series of Merton's optimum consumption-like problems, the optimal solution is explicitly derived. It is shown that if the prior benchmark level is too high (which can be verified), then the investment risk will be beyond *any* agent's risk tolerance. If the benchmark level is reasonable, then the optimal solution will be the same as that of one of the Merton's series problems, but with a particular value of absolute risk aversion, which is given by an explicit algebraic equation as a part of the optimal solution. According to our result, it is riskier to achieve the same surplus profit in a stable market than in a less-stable market, which is consistent with the common financial intuition.

1. Introduction

Diamond and Stiglitz (1974) point out that whether or not a person takes a gamble depends on two distinct considerations:

(i) The attributes of the gamble and, in particular, how risky it is; and
(ii) The attributes of the person and, in particular, how averse he or she is to risk.

In terms of the first issue, the concept of the risk measure has been used to explain how risky a gamble is. Many well-studied risk measures are described in the literature, such as the superhedging price, value at risk, tail value at risk, and expected shortfall as well as general coherent risk measures. These measures emphasize certain aspects of risk. However, few of them directly reflect the risk-averse person's attitude; that is, the perspective that "risk is what risk-averters hate" (Machina & Rothschild, 2008). The entropic risk measure, which depends on such a risk aversion through the exponential utility function, is one of the few to have attempted to capture this feature. In order to overcome the drawbacks of the existing measures, Aumann and Serrano (2008) have developed one risk measure which emphasizes such a risk-averters' attitude. This preserves many properties of the coherent risk measure such as first-order monotonicity, convexity and positive homogeneity. Unlike the coherent risk measure, however, it is also second-order monotonic, which is consistent with the emphasis on the risk-averters' attitude. Unfortunately, Aumann and Serrano (2008) only define the measure for a certain type of discrete random variables called gambles. It is acknowledged that most outcomes in financial applications are of continuous or mixed type, so their measure cannot be applied to many of theses outcomes. To incorporate general outcomes such as price of stocks, options and general contingent claims, this paper generalizes the definition of the measure to cover all random variables. The measure, like the original, will satisfy an essential axiom, namely the duality axiom. This axiom states that if one agent, who is uniformly more risk-averse than another agent, accepts a gamble, the latter will accept any less-risky gamble under the measure. It clearly demonstrates the solid connection between the measure and the attitude of the risk-averter. We therefore label it as the duality risk measure or duality index. The axiomatic characterization of the measure will be considered in detail in the following section.

In terms of the second consideration, utility functions have been used to describe the risk-aversion of an agent. The most widely

☆ The author thanks Ching, Wai-Ki at the Department of Mathematics in the University of Hong Kong for his kind referring of the celebrated work of Aumann and Serrano (2008). The author acknowledges financial support from Hong Kong Early Career Scheme (No. 533112), Hong Kong General Research Fund (No. 529711) and Hong Kong Polytechnic University.

* Tel.: +852 27666962.
E-mail address: maxu@polyu.edu.hk

http://dx.doi.org/10.1016/j.ejor.2014.06.022
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used utility functions are concave, which represents that the agent is globally risk-averse. Kahneman and Tversky (1979, 1992) consider S-shaped utility functions[1] to reflect the risk-seeking attitude of the agent in a loss situation and the risk-averse attitude in a gain situation. Meanwhile, they also introduce a reference point to separate gain and loss situations. Though many other utility functions are considered in the literature, only the globally risk-averse (which includes risk-neutral) agent as well as a reference point will be discussed in this paper. The reference point reflects the agent's relative financial situation.

To incorporate both considerations, namely how risky an outcome is and how risk-averse the agent is, we introduce a portfolio selection problem. This problem aims to find out a portfolio that minimizes the duality risk measure of the utility of the relative investment outcome, that is the difference between the investment outcome and the benchmark level. The risk measure addresses the first consideration and the utility function the second. Since the duality risk measure is highly nonlinear, we adopt a novel idea to deal with the portfolio selection problem, by firstly linking the problem to a series of Merton's optimum consumption-like problems, and then solving them using the well-known Lagrange method. It turns out that the original problem is equivalent to one of the series problems but with a particular choice of absolute risk aversion, which is given by an explicit algebraic equation as a part of the explicit optimal solution. Thus the explicit solution of the original problem is derived and the problem is completely solved. A critical threshold is also derived, so that once the surplus level (that is, the difference between the benchmark level and initial endowment) is beyond a threshold, the investment risk will exceed the agent's risk tolerance. In particular, if the agent is risk-neutral, that is to say with a linear utility function, then the investment risk will grow linearly with respect to (w.r.t.) the surplus level. The investment risk is also positively related to the entropy of the pricing kernel of the market. The result verifies the common financial intuition that it is much harder and riskier to achieve the same surplus profit in a stable market than in a less-stable market.

The paper is organized as follows. Section 2 defines the duality risk measure for all outcomes, studies its properties and axiomatic characterization, and then shows that it is the unique nontrivial index satisfying two axioms. Section 3 presents a portfolio selection problem under a complete market setting. The problem is to find a possible outcome to minimize the duality risk measure of the utility of the relative investment outcome. Section 4 is devoted to solving this portfolio selection problem. We first study how well posed the problem is, that is, whether its value is finite. Then we link it to a series of Merton's optimum consumption-like problems via a bridge problem. The series is then treated using the standard Lagrange method. Finally, the optimal solution and value of the original problem are derived. Analytical and numerical examples are also presented in Section 4 to illustrate the main result of this paper. We conclude the paper in Section 5.

2. Definition and characterization of the duality index

In order to define the duality risk measure, we need to review some of the concepts used in Aumann and Serrano (2008).

A gamble is a random variable whose mean is positive and that takes finitely many values, some of which are negative.

Say that an agent with utility function u accepts a gamble g at wealth w[2] if $\mathbf{E}[u(w+g)] > u(w)$, where $\mathbf{E}$ stands for "expectation"; that is to say, the agent prefers taking that gamble to refusing it at wealth w.[3] Throughout this paper, we only consider the risk-averse (which includes risk-neutral) agent; that is to say, u is concave.

Say that one agent is *uniformly more risk-averse* than another, if whenever the former accepts a gamble at some wealth, the latter accepts that gamble at any wealth, but not vice versa.

A *risk measure* or *index* is a (positive) real-valued function on gambles. A gamble is less risky than another under an index if its index value is strictly less than that of the latter.

Now we will introduce two important axioms related to indices.

Duality axiom: If one agent, who is *uniformly more risk-averse than* another, accepts a gamble, then the latter agent will accept any less-risky gamble under the index.
Positive homogeneity axiom: If a gamble is scaled by some positive scalar, then the index value is also scaled by the same scalar.

Aumann and Serrano (2008) show that, up to a positive multiple, there is a unique index satisfying the above two axioms. The duality axiom is more central than the other because together with the weak conditions of continuity and monotonicity it already implies that the index is unique up to the ordinal equivalent. Thus, we call the unique index satisfying both the duality and positive homogeneity axioms the *Aumann–Serrano duality risk measure* or *Aumann–Serrano duality index*, or simply the *duality risk measure* or *duality index*. Some important properties of the duality index are listed as follows.

Sub-additive: The duality index of the sum of two gambles is no more than the sum of the indices of each gamble.
Law-invariant: The duality indices of two identically distributed gambles are the same.
Convex: If a gamble is a linear combination of two gambles, then its duality index is no more than the same combination of the indices of each gamble.
Monotonic: The duality index decreases monotonically w.r.t. the first- and second-order (stochastic) dominance.[4]

It is noted that the convexity property is not stated explicitly by Aumann and Serrano (2008), however, this property will play a very important role in our analysis. It accords with the widely accepted financial wisdom that diversified investment reduces risk.

Now, let us define the duality index for general outcomes.

2.1. New definition of duality index

This paper is going to investigate a portfolio selection problem under the duality index. Portfolio selection means finding the best possible outcome in a certain set under a certain meaning, in the current setting, that is related to the duality index. Every

[1] S-shaped utility function is convex on the negative return and concave on the positive return.

[2] Throughout this paper, wealth is constant.

[3] By this definition, no agent accepts 0, which is inconsistent with financial intuition that no loss is an acceptable situation. In Aumann and Serrano (2008), it is not an issue because 0 is not regarded as a gamble. However, we will regard 0 as an outcome in this paper, so we assume that all agents accept 0 throughout this paper. It would not make any essential difference if 0 was assumed to be accepted by nobody.

[4] Say that one gamble *first-order dominates* another one, if its value is always no less than the latter. Say that one gamble *second-order dominates* another one, if the latter can be obtained by replacing some of the former's value with an outcome whose mean is that value. Say that one gamble *stochastically dominates* another one if there is a gamble distributed like the former that dominates the latter. A gamble g second-order stochastic dominates another gamble h if and only if $\mathbf{E}[f(g)] \leqslant \mathbf{E}[f(h)]$ for all decreasing and convex utility functions f.