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Decision Support

The stochastic ordering of mean-preserving transformations and its applications ☆

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ABSTRACT

The stochastic variability measures the degree of uncertainty for random demand and/or price in various operations problems. Its ordering property under mean-preserving transformation allows us to study the impact of demand/price uncertainty on the optimal decisions and the associated objective values. Based on Chebyshev's algebraic inequality, we provide a general framework for stochastic variability ordering under *any* mean-preserving transformation that can be parameterized by a single scalar, and apply it to a broad class of specific transformations, including the widely used mean-preserving affine transformation, truncation, and capping. The application to mean-preserving affine transformation rectifies an incorrect proof of an important result in the inventory literature, which has gone unnoticed for more than two decades. The application to mean-preserving truncation addresses inventory strategies in decentralized supply chains, and the application to mean-preserving capping sheds light on using option contracts for procurement risk management.

1. Introduction

In the study of a variety of operations problems in the presence of uncertainties (including demand, purchase price, and/or lead-time uncertainties), it is often of great interest to assess the impact of the degree of randomness, also known as stochastic variability. However, this could be a challenging task without an easily tractable way of ordering randomness. A useful approach commonly used in the literature is to employ a mean-preserving transformation of the random variable, which not only factors out the effect of the mean, but also parameterizes the variability of the random variable by a single scalar.

In this paper, we provide a general framework for stochastic variability ordering under *any* mean-preserving transformation that can be parameterized by a single scalar, and then apply it to a broad class of specific transformations, including the mean-preserving affine transformation, truncation, and capping. The application to mean-preserving affine transformation rectifies an incorrect proof of a fundamental result in Gerchak and Mossman (1992) that has gone unnoticed for more than two decades. The application to mean-preserving truncation addresses inventory strategies in decentralized supply chains, and the application to mean-preserving capping sheds light on using option contracts for procurement risk management.

The remainder of the paper is organized as follows: Section 2 reviews related literature. Section 3 provides a general framework for stochastic variability ordering under mean-preserving transformations, which is our main technical results. Section 4 discusses in great detail three different applications of our general framework. The paper concludes in Section 5.

2. Literature review

Stochastic variability plays a pivotal role in supply chain management because it directly affects the inventory policies and the associated total operational costs. Nyoman Pujawan (2004) studies the order variability in the upstream of the supply chain when the downstream uses different lot sizing rules: one to minimize the average cost per time and the other to minimize the average cost per unit. Garcia Salcedo, Ibeas Hernandez, Vilanova, and Herrera Cuartas (2013) studies a multi-echelon inventory system under both decentralized and centralized controls, and compares their impact on the demand variance amplification (the so-called bullwhip effect) from the end to the beginning of the supply chains. Both Nyoman Pujawan (2004) and Garcia Salcedo et al. (2013) use variance as the measure of stochastic variability. In general,

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however, variance is not a proper measure of risk (see Gerchak & Mossman (1992); an example is available on page 137 of Levy (2006)). A rigorous way to rank variability calls for theories of stochastic ordering.

The theoretical study of stochastic ordering can find its root in mathematics, and has drawn attention of researchers in operations engineering and economics. The underlying mathematical structure of stochastic ordering is driven by the concept of Majorization (Marshall, Olkin, & Arnold, 2010), which formalizes the intuitive notion that the components of a vector are more spread out than those of another vector. Many basic mathematical inequalities, including a special case of Chebyshev's inequality, are used to characterize the Majorization. A comprehensive collection of stochastic ordering properties and their applications in operations engineering can be found in Shaked and Shanthikumar (2007), which dedicates its chapter 3 to variability orders. The convex order of random variables discussed in that chapter is the concept we adopt in this paper. The convex order is closely related to the stochastic dominance, which is a subject studied in economics for decision making under uncertainty (Levy, 2006). In particular, the second order stochastic dominance is equivalent to the convex order when two random variables have equal means. Most of the above-mentioned literature study general stochastic ordering properties. Marshall et al. (2010) is more akin to our paper as it studies families of scalar parameterized distributions, but it limits its attention to a few specific distributions, e.g., exponential, whereas our paper develops a general framework to parameterize stochastic variability ordering for any distribution.

The study of the mean-preserving transformation first appears in Rothschild and Stiglitz (1970), which shows that a riskier (stochastically more variable) random variable can be constructed by adding a white noise of zero mean to the original random variable. However, the limitation of this mean-preserving transformation is the lack of a convenient means to assess the impact of stochastic variability. Gerchak and Mossman (1992) overcomes this limitation by constructing a mean-preserving *affine* transformation that can be parameterized by a single scalar. It performs a mean-preserving transformation of a random variable in a linear fashion, and shows that the stochastic order of the transformed variables is monotone in the parameter. The same affine mean-preserving transformation is also used to study the value of inventory pooling in Gerchak and He (2003).

Our paper differs in that we provide a general framework, which not only covers the mean-preserving affine transformation, but also extends to other types of transformations, i.e., nonlinear transformations. These nonlinear mean-preserving transformations of random variables are commonly seen in operations applications. For example, under a scheduled ordering scheme in a supply chain (Chen & Gavirneni, 2010), the demand distribution in the upstream is transformed from the truncation of the downstream demand by a pre-agreed fixed shipment quantity. In procurement with options (Nagali et al., 2008), the purchase price distribution is transformed from the uncertain market price, with a cap on the strike price specified in the option contract. Our unified framework enables us to use a single scalar as the parameter to study the stochastic variability ordering properties of all the aforementioned mean-preserving transformations.

3. A general framework for stochastic variability ordering under mean-preserving transformations

The general framework makes use of a generalized Chebyshev's Algebraic Inequality (Mitrinović, Pečarić, & Fink, 1993), which is replicated in Lemma 1 below for ease of reference. Note that throughout the text, the terms increasing and decreasing should be taken in their weak sense; e.g., increasing means nondecreasing.

Lemma 1 (Mitrinović et al. (1993), p. 248). *Let $u, g : [a,b] \to \mathbb{R}$ and $F : [a,b] \to [0,1]$ be a distribution function. Suppose that u is monotonically increasing. Define $G_F : (a,b] \to \mathbb{R}, G_F(t) = \int_a^t g(s)dF(s)/\int_a^t dF(s)$. If*

$$G_F(t) \leqslant G_F(b) \tag{1}$$

for all $t \in (a,b]$, then

$$\int_a^b u(s)g(s)dF(s) \geqslant \int_a^b u(s)dF(s) \int_a^b g(s)dF(s) \tag{2}$$

If (1) holds and u is monotonically decreasing, the reverse of (2) holds. In addition, as a special case, (2) holds if $g(s)$ is increasing.

We are now ready to present our main result below in Theorem 1, which identifies unifying sufficient conditions for using a scalar parameter to order the stochastic variability of random variables under *any* mean-preserving transformation that can be parameterized by a single scalar.

Let X be a nonnegative random variable defined on the support $[a,b]$ with cumulative distribution function $F : [a,b] \to [0,1], a \geqslant 0$, and $Y(X,\alpha)$ be a mean-preserving transformation of X, i.e., for all $\alpha \geqslant 0$,

$$E[Y(X,\alpha)] = E[X] = \mu, \tag{3}$$

where $Y(X,\alpha)$ is assumed to be piecewise differentiable. As we shall see below in Theorem 1, the stochastic variability of $Y(X,\alpha)$ is parameterized by the single scalar α.

In the following text, for notational convenience, we use a subscript to specify the range of the integration whenever an expectation is not taken on the entire support of a random variable. For instance, we use $E_{X \leqslant t}[\partial Y(X,\alpha)/\partial\alpha]$ to denote $\int_a^t \partial Y(X,\alpha)/\partial\alpha dF(x)$.

Theorem 1. *As α increases, $Y(X,\alpha)$ becomes stochastically more variable if the following conditions hold:*

(a) $Y(X,\alpha)$ is increasing in X;
(b) for any $a \leqslant t \leqslant b, E_{X \leqslant t}[\partial Y(X,\alpha)/\partial\alpha] \leqslant 0$.

Proof. It is well known that $Y(X,\alpha_1) \geqslant_v Y(X,\alpha_2)$ if and only if $E[h(Y(X,\alpha_1))] \geqslant E[h(Y(X,\alpha_2))]$ for any convex function h (see Theorem D.1(c) of Porteus (2002)), where the operator $\geqslant_v$ denotes second order stochastic dominance, i.e., $Y(X,\alpha_1) \geqslant_v Y(X,\alpha_2)$ means $Y(X,\alpha_1)$ is stochastically more variable than $Y(X,\alpha_2)$. Therefore, it suffices to show that for any convex function $h, E[h(Y(X,\alpha))]$ is increasing in α, i.e., $\partial E[h(Y(X,\alpha))]/\partial\alpha \geqslant 0$, under the two conditions stated in the theorem.

By the chain rule,

$$\frac{\partial E[h(Y(X,\alpha))]}{\partial\alpha} = E\left[\frac{\partial h(Y(X,\alpha))}{\partial Y(X,\alpha)}\frac{\partial Y(X,\alpha)}{\partial\alpha}\right] \geqslant E\left[\frac{\partial h(Y(X,\alpha))}{\partial Y(X,\alpha)}\right]E\left[\frac{\partial Y(X,\alpha)}{\partial\alpha}\right] = 0, \tag{4}$$

where the last step follows from the fact that $E[\partial Y(X,\alpha)/\partial\alpha] = \partial E[Y(X,\alpha)]/\partial\alpha = 0$, because $E[Y(X,\alpha)] = \mu$ is independent of α. The inequality (4) follows from Lemma 1, which is applicable because if we let $\partial h(Y(X,\alpha))/\partial Y(X,\alpha)$ and $\partial Y(X,\alpha)/\partial\alpha$ correspond to functions u and g respectively in Lemma 1, then the two sufficient conditions in Lemma 1 are implied by (a) and (b) of this theorem. In particular, the first condition of Lemma 1 requires $\partial h(Y(X,\alpha))/\partial Y(X,\alpha)$ to be monotonically increasing in X, which is true because h is convex and $Y(X,\alpha)$ is increasing in X (condition (a)); the second condition of Lemma 1 is $\int_a^t [\partial Y(x,\alpha)/\partial\alpha] dF(x)/F(t) \leqslant E[\partial Y(X,\alpha)/\partial\alpha]$, which is implied by (b) and the aforementioned fact that $E[\partial Y(X,\alpha)/\partial\alpha] = 0$. □