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Decision Support

Decision making under uncertainty with unknown utility function and rank-ordered probabilities

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ABSTRACT

We consider the ranking of decision alternatives in decision analysis problems under uncertainty, under very weak assumptions about the type of utility function and information about the probabilities of the states of nature. Namely, the following two assumptions are required for the suggested method: the utility function is in the class of increasing continuous functions, and the probabilities of the states of nature are rank-ordered. We develop a simple analytical method for the partial ranking of decision alternatives under the stated assumptions. This method does not require solving optimization programs and is free of the rounding errors.

1. Introduction

Within the framework of expected utility theory an alternative x' is preferred to alternative x'' if the expected utility of x' is greater than the expected utility of x'' (see, e.g., Fishburn, 1970). In practice, however, the utility function may be difficult to assess, and the probability distribution may not be completely known. In particular note that the assumption of an additive utility function is a fairly strong assumption to make (Allais, 1953).

There exists extensive literature that deals with different relaxations of the above standard assumptions. For example, the literature on stochastic dominance (Whitmore and Findlay, 1978; Levy, 2006; Post and Kopa, 2013) generally assumes that the utility function is unknown and belongs to a certain class of functions, while the probability distribution on the set of states of nature is known. The opposite case in which the utility function is known but the information about probabilities is incomplete, is considered, for example, in Fishburn (1964, 1965), Hazen (1986), and Parkan (1994).

A more difficult case arises when both the utility function is not known exactly (but a class of functions to which it belongs is specified) and the information about probabilities is incomplete. Based on this assumption, the ranking of decision alternatives requires solving nonlinear optimization problems (Weber, 1987; Moskowitz et al., 1993). Pearman and Kmietowicz (1986) consider a special case in which the probabilities are described by sets of linear inequalities, and the utility function is assumed either monotone or monotone and concave. In both cases the comparison of decision alternatives leads to the solution of linear programs. Keppe and Weber (1990) show that the use of so-called P-matrices reduces the number of linear programs that need to be solved. In special cases of the latter paper, the solution of linear programs is not even required.

Methodologically close to the above description are multi-criteria decision problems (under certainty) in which the weights of criteria (coefficients of importance) and/or criteria values for different alternatives are not known precisely (Podinovski, 2004). In particular, some methods developed for the former problems are applicable to the (single-criterion) decision problems under uncertainty considered in this paper. This includes methods developed for problems with homogeneous criteria – by definition the latter have a common scale. Examples of such methods that are based on a further assumption of an additive value function include analytical approaches of Kirkwood and Sarin (1985) and Corrizosa et al. (1995). Further examples of methods equally applicable to both types of problem include methods of criteria importance theory (Podinovski, 1993, 2002) – and these do not assume the existence of a value function. Note however that not all methods are transferable between the two types of problem. For example, the optimizing method of Eum et al. (2001) is not applicable to the decision problems considered in our paper.

In our paper we contribute to the literature by considering the case based on a very weak assumption on the utility function and a further assumption that the probabilities of the states of nature are ordered by their value. We develop a simple analytical method that allows the partial ranking of decision alternatives under the stated

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assumptions. This method does not require the solution of optimization programs.

The paper is organized as follows. Section 2 introduces the basic definitions and notation. Section 3 contains the main result of our paper – a method for pairwise comparison of alternatives. Section 4 presents an illustrative application. The concluding Section 5 summarizes our development. All proofs are given in Appendix A.

2. Basic definitions

We use the following notation:

X – the set of decision alternatives
$S = \{s^1, \dots, s^n\}$ – the set of states of nature
$\eta(x,s^i)$ – the consequence (pay-off) to the decision maker defined by alternative x and state s^i (we may view η is a function whose domain is $X \times S$ and codomain is H – the latter is the set of all possible consequences)
$p = (p_1, \dots, p_n)$ – the probability distribution on S: p_i is the probability that s^i is the true state of nature
$u(\eta(x,s^i))$ – the utility of the consequence $\eta(x,s^i)$.

For simplicity, instead of the vector of utilities $(u(\eta(x,s^1)), \dots, u(\eta(x,s^n))$, we use notation $u(x) = (u_1(x), \dots, u_n(x))$, where $u_i(x) = u(\eta(x,s^i))$.

If both the probability distribution p and utility function u are known, alternatives are ranked according to their expected utility. For an alternative x, the latter is defined as

$$\bar{u}(x) = \sum_{i=1}^{i=n} p_i u_i(x). \tag{1}$$

Below we assume that the probability distribution p is not known exactly. Instead, we assume that the components of vector p satisfy the following inequalities:

$$p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0. \tag{2}$$

Let each non-strict inequality in (2) be either a strict inequality or equality, and let us specifically know which of the two relations is true. For example, we know that either $p_1 > p_2$ or $p_1 = p_2$.

Let $\overrightarrow{\Pi}$ denote the set of all vectors p that satisfy the stated ordinal information (2). (We also make the standard assumption that all $p_i > 0$ and $p_1 + \cdots + p_n = 1$.)

We further assume that the unknown utility function is of the class U^1 of all increasing continuous functions. Following a well-established approach (Weber, 1987; Greco et al., 2008), define the binary relations of non-strict preference R, (strict) preference P, and indifference I, on the set of all alternatives X as follows:

$$x'Rx'' \quad \text{if } \bar{u}(x') \geqslant \bar{u}(x'') \text{ is true for all } p \in \overrightarrow{\Pi} \text{ and } u \in U^1; \tag{3}$$
$$x'Px'' \quad \text{if the relation } x'Rx'' \text{ is true but } x''Rx' \text{ is not true};$$
$$x'Ix'' \quad \text{if both relations } x'Rx'' \text{ and } x''Rx' \text{ are true}.$$

3. The main result: comparing the alternatives

Consider the set $\overrightarrow{\Pi}$ of all probability vectors p that satisfy (2). Let N_1 be the set of all states i that have the highest probability p_i of occurrence as stated in (2). (Because in (2) we allow equalities, there may be more than one state in this set.) Let i_1 be the largest i such that $i \in N_1$. Let N_2 be the set of states that have the second highest probability p_2, and i_2 the largest i such that $i \in N_2$. Continuing this process, finally let N_ρ be the set of states i that have the lowest probability among all states, and let i_ρ be the largest i such that $i \in N_\rho$.

For example, let

$$p_1 > p_2 = p_3 > p_4. \tag{4}$$

Then $N_1 = \{1\}$, $i_1 = 1$, $N_2 = \{2,3\}$, $i_2 = 3$, $N_3 = \{4\}$, $i_3 = 4$, and $\rho = 3$.

Suppose we want to compare alternatives x' and x'' as defined in (3). Define the corresponding vectors of pay-offs

$$a = (a_1, \dots, a_n), \quad \text{where } a_i = \eta(x', s^i), \quad i = 1, \dots, n;$$
$$b = (b_1, \dots, b_n), \quad \text{where } b_i = \eta(x'', s^i), \quad i = 1, \dots, n.$$

Furthermore, for any vector $c = (c_1, \dots, c_n)$, let c^j be the vector consisting of the first j components of vector c:

$$c^j = (c_1, \dots, c_j).$$

Let $c^j_\downarrow = \left(c^j_{[1]}, \dots, c^j_{[j]}\right)$ denote the vector obtained by permuting the components of vector c^j in the non-increasing order: $c^j_{[1]} \geqslant \cdots \geqslant c^j_{[j]}$. For example, if $c = (1,4,3)$, then $c^2 = (1,4)$ and $c^2_\downarrow = (4,1)$.

Finally, for any two vectors $(c_1, \dots, c_m)$ and $d = (d_1, \dots, d_m)$, the vector inequality $c \geqslant d$ is used to denote the set of inequalities between the corresponding components: $c_1 \geqslant d_1, \dots, c_m \geqslant d_m$.

The following theorem establishes a constructive method for the comparison of alternatives x' and x'' as defined in (3).

Theorem 1. *The relation $x'Rx''$ is true if and only if the following inequalities are true:*

$$a^{i_r}_\downarrow \geqslant b^{i_r}_\downarrow, \quad r = 1, \dots, \rho. \tag{5}$$

Furthermore, if in (5) all non-strict inequalities are satisfied as equalities, then $x'Ix''$, otherwise $x'Px''$.

The proof of Theorem 1 is given in Appendix A.

Remark 1. If all states of nature are equally probable, then $\rho = 1$, $i_1 = n$, and (5) is reduced to the single inequality $a_\downarrow \geqslant b_\downarrow$, where $a_\downarrow = a^n_\downarrow$ and $b_\downarrow = b^n_\downarrow$. This is equivalent to the result obtained in Podinovski (1975) stated in terms of multi-criteria problems. In particular, Podinovski (1975) showed that the corresponding preference relation R arising in multi-criteria problems could be regarded as the ordinal formulation of the principle of insufficient reason for decision making under ignorance (Luce and Raiffa, 1957).

Example 1. Let $n = 4$ and p satisfy (4). Then $i_1 = 1$, $i_2 = 3$, $i_3 = 4$, and $\rho = 3$. Consider three alternatives x^1, x^2 and x^3 whose values $\eta(x,s)$ are shown in Table 1.

To verify relation x^1Rx^2, we use Theorem 1. For $x' = x^1$ and $x'' = x^2$ we have:

$$a^1 = 4, \quad a^1_\downarrow = 4; \quad b^1 = 3, \quad b^1_\downarrow = 3;$$
$$a^3 = (4,3,4), \quad a^3_\downarrow = (4,4,3); \quad b^3 = (3,2,7), \quad b^3_\downarrow = (7,3,2);$$
$$a^4 = (4,3,4,8), \quad a^4_\downarrow = (8,4,4,3);$$
$$b^4 = (3,2,7,2), \quad b^4_\downarrow = (7,3,2,2).$$

Then inequalities (5) take on the form:

$$a^1_{[1]} \geqslant b^1_{[1]}, \quad \text{i.e. } 4 \geqslant 3;$$
$$a^3_{[1]} \geqslant b^3_{[1]}, \quad a^3_{[2]} \geqslant b^3_{[2]}, \quad a^3_{[3]} \geqslant b^3_{[3]}, \quad \text{i.e. } 4 \geqslant 7, \quad 4 \geqslant 3, \quad 3 \geqslant 2;$$
$$a^4_{[1]} \geqslant b^4_{[1]}, \quad a^4_{[2]} \geqslant b^4_{[2]}, \quad a^4_{[3]} \geqslant b^4_{[3]}, \quad a^4_{[4]} \geqslant b^4_{[4]}, \quad \text{i.e. } 8 \geqslant 7,$$
$$4 \geqslant 3, \quad 4 \geqslant 2, \quad 3 \geqslant 2.$$

Not all of the above inequalities are true and, by Theorem 1, the relation x^1Rx^2 does not hold. Similarly, x^2Rx^1 does not hold. Therefore, alternatives x^1 and x^2 are incomparable with respect to R.