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Decision Support

On the restricted cores and the bounded core of games on distributive lattices

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ABSTRACT

A game with precedence constraints is a TU game with restricted cooperation, where the set of feasible coalitions is a distributive lattice, hence generated by a partial order on the set of players. Its core may be unbounded, and the bounded core, which is the union of all bounded faces of the core, proves to be a useful solution concept in the framework of games with precedence constraints. Replacing the inequalities that define the core by equations for a collection of coalitions results in a face of the core. A collection of coalitions is called normal if its resulting face is bounded. The bounded core is the union of all faces corresponding to minimal normal collections. We show that two faces corresponding to distinct normal collections may be distinct. Moreover, we prove that for superadditive games and convex games only intersecting and nested minimal collection, respectively, are necessary. Finally, it is shown that the faces corresponding to pairwise distinct nested normal collections may be pairwise distinct, and we provide a means to generate all such collections.

1. Introduction

In cooperative game theory, for a given set of players N, TU games are functions $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, which express for each nonempty coalition $S \subseteq N$ of players the best they can achieve by cooperation. In the classical setting, every coalition may form without any restriction, i.e., the domain of v is indeed 2^N. In practice, this assumption is often unrealistic since some coalitions may not be feasible for various reasons, e.g., players may be political parties with divergent opinions or restricted communication abilities, or a hierarchy may exist among players and the formation of coalitions must respect the hierarchy, etc.

Many studies have been done on games defined on specific subdomains of 2^N, e.g., antimatroids (Algaba, Bilbao, van den Brink, & Jiménez-Losada, 2004), convex geometries (Bilbao, 1998; Bilbao, Lebrón, & Jiménez, 1999), distributive lattices (Faigle & Kern, 1992), or other structures (Béal, Rémila, & Solal, 2010; Faigle, Grabisch, & Heyne, 2010; Pulido & Sánchez-Soriano, 2006). In this paper, we focus on the case of distributive lattices. To this end, we assume that there exists some partial order $\preceq$ on N describing some hierarchy or precedence constraint among players, as in Faigle and Kern (1992). We say that a coalition S is feasible if the coalition contains all its subordinates, i.e., $i \in S$ implies that any $j \preceq i$ belongs to S as well. Then by Birkhoff's theorem, feasible coalitions form a distributive lattice.

The main problem in cooperative game theory is to define a reasonable solution of the game, that is, supposing that the grand coalition N will form, how to share among its members the total worth $v(N)$. The core (Gillies, 1959) is the most popular solution concept, since it ensures stability of the game in the sense that no coalition has an incentive to deviate from the grand coalition. For classical TU games, the core is either empty or a convex bounded polyhedron. However, for games whose cooperation is restricted, the study of the core is much more complex, since it may be unbounded or even contain no vertices (see a survey by Grabisch (2009)). For the case of games with precedence constraints, it is known that the core is *always* unbounded or empty but contains no line (i.e., it has vertices).

Unboundedness of the core induces difficulties in using it as a solution concept because, on the practical side, one cannot handle payment vectors that grow beyond any border. Moreover, from the mathematical point of view, the core is not compact, and this property is often required for establishing results. For example, a sequence of elements in the core, created by some negotiation procedure, may not have a convergent subsequence, so that the procedure does not help to finally select an element of the core.

Certainly there exist many ways of defining a compact subset of the core, e.g., one may take the convex hull of its vertices. Here, we choose another solution, called the *bounded core* (Grabisch & Sudhölter, 2012), which has a natural interpretation for games with

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precedence constraints. Indeed, the bounded core is the set of core elements such that every player takes the maximum of her direct subordinates, in the sense that any transfer from a subordinate to her boss would result in a payoff vector outside the core. Also, from a geometric point of view, the bounded core is the union of all bounded faces of the core.

Besides, bounded faces of the core have been studied by Grabisch (2011) under the name *restricted cores*. Bounded faces arise by turning some inequalities $x(S) \geqslant v(S)$ of the core into equalities, so that the resulting face does not contain any extremal ray. From a game theoretic point of view, these additional equalities can be seen as binding constraints for certain coalitions, and hence the arising face is named *restricted* core. If the collection of coalitions with a binding constraint does induce boundedness of the resulting face, it is called a *normal collection*. In Grabisch (2011), some examples of normal collections are provided, and their properties are studied.

The aim of this paper is to investigate the structure of the bounded core with the help of normal collections. Specifically, we want to address the following combinatorial problem: The bounded core is the union of all bounded faces and, hence, it is the union of restricted cores with respect to all possible normal collections. However, the number of normal collections is huge, and we do not know any efficient way to generate them. Hence, the main question is: How can the bounded core be written as a union of a minimal number of faces? The second question naturally follows: How can the corresponding normal collections be generated?

We provide complete answers to these questions for the case of convex games and answer the first question in the case of superadditive games as well as for the general case. We establish that for the general case only minimal (in the size of the collection) normal collections are necessary and, moreover, each minimal normal collection is necessary in the sense that for each minimal normal collection $\mathcal{N}$, there exists a game such that there is a point in the bounded face induced by $\mathcal{N}$, which does not belong to any other bounded face (Proposition 6). In a similar result for superadditive games, we show that only intersecting minimal normal collections are needed (Proposition 7).

For convex games Theorem 5 shows that only nested minimal normal collections are needed. In this case it is proved that generically all faces that correspond to the nested minimal normal collections are needed in the following sense: For any strictly convex game the face corresponding to an arbitrary nested minimal normal collection contains an element that is not contained in a face that corresponds to any other nested minimal normal collection. Finally, we show that nested minimal normal collections can be generated by a special class of linear extensions of the partial order $\preceq$ on N. Besides, we show a generalization of the well-known Shapley-Ichiishi theorem for games with precedence constraints.

The paper is organized as follows. Section 2 establishes the basic material for the rest of the paper, and it presents the notions of restricted core, normal collection and bounded core. Section 3 studies the set of normal collections, introduces properties and recalls and discusses well-known examples of minimal normal collections. It also shows how nested collections can be obtained by a closure operator on a certain class of normal collections. Section 4 investigates the general case and the case of superadditive games. It also generalizes the Bondareva-Shapley theorem (Bondareva, 1963; Shapley, 1971) by suitably generalizing the balancedness conditions that are equivalent to the nonemptiness of bounded faces of the core. Section 5 investigates in depth the case of convex games, showing the fundamental role played by minimal nested normal collections.

2. Notation, definitions and preliminaries

Let $(P, \preceq)$ be a finite partially ordered set (*poset* for short), that is, a finite set P endowed with a reflexive, antisymmetric, and transitive relation (see, e.g., Davey & Priestley, 1990). We denote by $\prec$ the asymmetric part of $\preceq$. We say that $x \in P$ *covers* $y \in P$, and we denote it by $y \prec\cdot x$ if $y \prec x$ and there is no $z \in P$ such that $y \prec z \prec x$.

We denote by $\min(P)$ and $\max(P)$, respectively, the set of the minimal and maximal elements of $(P, \preceq)$. The *dual* of the poset $(P, \preceq)$, denoted by $(P, \preceq^\partial)$ (or simply P^∂), is the set P endowed with the reverse order, i.e., $x \preceq y$ if and only if $y \preceq^\partial x$.

Throughout the paper, it is understood that any subset Q of a poset $(P, \preceq)$ is endowed with $\preceq$ restricted to Q (we do not use a special symbol for the restriction).

A *chain* C is a subset of P such that its elements are pairwise comparable, i.e., for any two elements $x, y \in C$, we have $x \preceq y$ or $y \preceq x$, whereas an *antichain* is a subset of pairwise incomparable elements of P. A chain C is *maximal* if no other chain contains it or, equivalently, if $C = \{x_1, \ldots, x_q\}$, with $x_1 \prec\cdot x_2 \prec\cdot \cdots \prec\cdot x_q$ and $x_1 \in \min(P)$, $x_q \in \max(P)$. Its *length* is $q - 1$. The *height* of $i \in P$, denoted by $h(i)$, is the length of a longest chain from a minimal element to i. Elements of same height k form *level k*, denoted by L_k. Hence, $L_0 = \min(P)$ is the set of all minimal elements, $L_1 = \min(P \setminus L_0)$, $L_2 = \min(P \setminus (L_0 \cup L_1))$, etc. The *height* of N, denoted by $h(N)$, is the maximum of $h(i)$ taken over all elements of N. Similarly, we define the *depth* $d(i)$ of an element $i \in N$ as its height in the dual poset P^∂. We denote by D_0 the set of all elements of depth 0, and we have that $D_0 = \max(P)$, $D_1 = \max(P \setminus D_0)$, $D_2 = \max(P \setminus (D_0 \cup D_1))$, etc.

A *lattice* is a poset $(L, \preceq)$, where for each $x, y \in L$ their supremum $x \vee y$ and infimum $x \wedge y$ exist. The lattice is distributive if $\vee, \wedge$ obey distributivity.

A subset $Q \subseteq P$ is a *downset* of P if $x \in Q$ and $y \preceq x$ implies $y \in Q$. We denote by $\mathcal{O}(P, \preceq)$ the set of downsets of $(P, \preceq)$. It is a well-known fact that $(\mathcal{O}(P, \preceq), \subseteq)$ is a distributive lattice and every distributive lattice arises that way (Birkhoff, 1933). We denote by $\downarrow x$ the downset generated by an element $x \in P$, that is, $\downarrow x = \{y \in P | y \preceq x\}$. Similarly, for any $Q \subseteq P$, $\downarrow Q = \bigcup_{x \in Q} \downarrow x$.

Let N be a finite set of n players. A *set system* $\mathcal{F}$ on N is a collection of subsets of N containing N and $\emptyset$. Any nonempty subset in $\mathcal{F}$ is called a *feasible coalition*. We define a *cooperative TU game with restricted cooperation* (or simply a *game*) on $\mathcal{F}$ as the pair $(\mathcal{F}, v)$, with $v : \mathcal{F} \to \mathbb{R}$, such that $v(\emptyset) = 0$.

In this paper we focus on a particular case of set systems, introduced by Faigle and Kern (1992) (games with precedence constraints). We consider a partial order $\preceq$ on N, which may express precedence constraints among players, or hierarchical relations. A coalition S is *feasible* if whenever $i \in S$, all subordinates of i also belong to S, i.e., S is a downset of $(N, \preceq)$. In other words, $\mathcal{F} = \mathcal{O}(N, \preceq)$, and hence $\mathcal{F}$, partially ordered by inclusion, is a distributive lattice, where supremum and infimum are, respectively, $\cup, \cap$. In the sequel we often omit braces for singletons, writing, e.g., 1^i instead of $1^{\{i\}}$.

A game $(\mathcal{F}, v)$ with $\mathcal{F} = \mathcal{O}(N, \preceq)$ is *convex* if

$$v(S \cup T) + v(S \cap T) \geqslant v(S) + v(T) \text{ for all } S, T \in \mathcal{F}. \quad (1)$$

It is *superadditive* if the above inequalities are valid for disjoint sets S, T. It is *strictly convex* if the inequalities (1) are strict for $S \setminus T \neq \emptyset \neq T \setminus S$.

The following lemma extends a classical result when $\mathcal{F} = 2^N$.

Lemma 1. *Let $\mathcal{F} = \mathcal{O}(N, \preceq)$ and $(\mathcal{F}, v)$ be a game. Then $(\mathcal{F}, v)$ is convex if and only if for all $i \in N$,*

$$v(P \cup i) - v(P) \leqslant v(Q \cup i) - v(Q) \text{ for all } P \subsetneq Q \subseteq N \setminus i \text{ with } P \cup i, Q \in \mathcal{F}, \quad (2)$$