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Interfaces with Other Disciplines

Maintaining the Regular Ultra Passum Law in data envelopment analysis

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ABSTRACT

The variable returns to scale data envelopment analysis (DEA) model is developed with a maintained hypothesis of convexity in input–output space. This hypothesis is not consistent with standard microeconomic production theory that posits an S-shape for the production frontier, i.e. for production technologies that obey the Regular Ultra Passum Law. Consequently, measures of technical efficiency assuming convexity are biased downward. In this paper, we provide a more general DEA model that allows the S-shape.

1. Introduction

The non-parametric data envelopment analysis (DEA) approach envelops observed data with a piecewise linear frontier. The characteristics of a DEA model are derived from a number of maintained assumptions imposed on the technology. A typical estimator used in DEA is the BCC-estimator (Banker, Charnes, & Cooper, 1984), which assumes the estimated production possibility set is a polyhedral set that allows variable returns to scale. As a consequence, the BCC-estimator assumes that marginal product is non-increasing, which violates standard microeconomic theory where marginal product initially increases but diminishing returns eventually set in. In particular, if data reflects the Regular Ultra Passum (RUP) law (Frisch, 1965, Chapter 8), the BCC-estimator will be biased downward.

Definition 1. The RUP law. Let a single output y be produced from a vector of m inputs x according to a production function $F(x,y)=0$. This production function obeys the RUP law if $\frac{\partial \varepsilon(x,y)}{\partial x_i}<0,\ i=1,\ldots,m$ where the function $\varepsilon(x,y)$ is the scale elasticity, and for some point (x_1,y_1) we have $\varepsilon(x_1,y_1)>1$, and for some point (x_2,y_2), where $x_2>x_1,\ y_2>y_1$, we have $\varepsilon(x_2,y_2)<1$.

The problem with the BCC-estimator is that the supporting hyperplanes for envelopment can overestimate inefficiency for points that should be projected to the local non-convex segments of the true frontier characterized by increasing returns to scale.[1] In this paper, we are concerned with production technologies satisfying the RUP condition where the BCC-estimator is biased because such technologies are not convex in input–output space.[2] Furthermore, existing measures of scale efficiency will be biased due to the improper projection to production impossibilities. The main contribution of this paper is the development of an approach that is capable of measuring inefficiencies for production possibilities in a non-convex homothetic and S-shaped technology. A non-convex S-shaped technology is characterized as follows: along any expansion path an expanding DMU with low activity will have a high scale elasticity greater than one. As the unit expands its activity the scale elasticity will decrease and will approach optimal scale size with an elasticity equal to one. Further expansion will imply

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[1] See Fig. 2 later in the paper for a geometric comparison of the BCC- and the proposed S-shaped estimator.

[2] It is argued in Banker et al. (1984) that any point located in the interior of a strongly efficient facet with a supporting hyperplane given by $\{(x,y)|u^ty-v^tx-u_o=0\}$ will have the local scale characteristics determined from the sign of u_o. Hence, as stated in (29a), page 1087 in Banker et al. (1984) we have "*Increasing returns to scale* $\iff u_o<0$". In other words, the convex hull estimator proposed in the BCC model will in general accommodate estimation of local scale characteristics of both increasing, constant and decreasing returns to scale. However, if the true production function satisfies the RUP law with monotonically decreasing scale-elasticity then the convex hull estimator may provide biased efficiency scores for observations below most productive scale size (mpss). The possible radial contraction of the input vectors from such observations towards the boundary of the convex hull estimator may provide exaggerated estimates of possible input contractions. This could, e.g., happen if such an observation is dominated by a point on a facet spanned entirely by observations below mpss, where some observations are close to the origin and all other observations are close to being mpss.

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decreasing returns with a scale elasticity less than one and approaching zero.[3]

A twice differentiable *nicely convex-concave* production function $h(x)$ in the terminology suggested by Ginsberg (1974) is an example of an S-shaped technology for the single input single output case. A nicely convex-concave production function satisfies the following assumptions (i) $h(0)=0$, (ii) $h(x)\geqslant 0,\ x\in[0,\infty)$, (iii) $h'(x)>0,\ x\in(0,\infty)$, (iv) there exist $x^*\in(0,\infty)$ such that $h''(x)>0,\ x\in(0,x^*)$ and $h''(x)<0,\ x\in(x^*,\infty)$, (v) there exists a $\bar{x}, x^*<\bar{x}<\infty$, such that $h(\bar{x})=\bar{x}\times h'(\bar{x})$. By reference to Ginsberg's PhD thesis it is argued in Ginsberg (1974) that a nicely convex-concave production function will have an average product being non-negative and increasing until it reaches its maximum at $\bar{x}$. For $x>\bar{x}$ the average product will decrease for increasing x. It is easy to prove that this implies that the scale elasticity is monotonically decreasing, i.e. the production function satisfies the RUP law.[4]

Several non-convex models exist in the literature (e.g., the FDH-model of Afriat (1972), Deprins, Simar, and Tulkens (1984), the Petersen-Bogetoft approach, Petersen (1990), Bogetoft (1996)), and Jeong and Simar (2006), Kuosmanen (2001) but these models are not well-suited to estimate an S-shaped production structure because any non-convex shape can result from these estimation procedures. In other words, we are looking for an estimation procedure that allows only non-convexities that are reflected in an S-shaped production structure.[5] For simplicity, we focus on production technologies that are homothetic. The concept of a homothetic production function was first introduced in Shephard (1953, page 30) as a monotonic transformation of a linear homogenous production function. With a homothetic production structure we can smooth the obtained structure of the estimated isoquant because homotheticity implies that the shape of the isoquants is identical. This allows us to maintain convexity in input (and output space) and to allow non-convexities in input–output space.

In order to move between input space and output space, we propose estimating individual isoquants assuming selective input convexity using a simplified order-m estimation procedure (Cazals, Florens, & Simar, 2002) where we avoid replications. The order-m estimation procedures include a conditional estimation model maintaining selective convexity of the input sets.[6] Under the assumption of homotheticity, we can aggregate inputs (and outputs) allowing us to move to aggregate input–output space where we can impose an S-shape.

The rest of the paper is organized as follows. In Section 2 we define the production technology, from an input orientation using an input distance function. The assumption of homotheticity is presented and the implication for input aggregation is discussed. Notably, the assumption of homotheticity allows us to generate any isoquant from a base isoquant and hence, derive a well-defined index of aggregate input. Section 3 is devoted to the estimation of the base isoquant using a conditional estimator. We also discuss criteria for selecting a well-estimated isoquant among all possible base isoquants to aggregate inputs. This isoquant is used for the aggregation of inputs. In Section 4, we develop a model to estimate a piecewise linear S-shaped frontier. Using simulated data in Section 5, we show that our method overcomes the inherent problems of standard DEA and provides better estimates of inefficiency when the true technology obeys the Regular Ultra Passum Law. The last section concludes with directions for future research.

2. Production technology

Let us consider a production environment where a vector of s inputs $X=(x_1,\ldots,x_s)$ is used in the production of one output Y. We represent the production technology with the input set $L(Y)=\{X\in\mathbb{R}^s_+ : X \text{ can produce } Y\}$ which has isoquant

$$IsoqL(Y)=\{X : X\in L(Y), \lambda X\notin L(Y), \lambda\in[0,1)\} \tag{1}$$

Since we assume that only one output is produced, we can define a production function as

$$\phi(X)=\max\{Y : X\in L(Y)\} \tag{2}$$

The input distance function (Shephard, 1970) is then defined as

$$D_I(Y,X)=\max\{\gamma : X/\gamma\in L(Y)\} \tag{3}$$

which provides an alternative characterization of the technology since $D_I(Y,X)\geqslant 1 \iff X\in L(Y)$. Finally, the index of technical efficiency proposed by Debreu (1959) and Farrell (1957) that serves as basis for DEA is given as

$$F_I(Y,X)=\min\{\gamma : \gamma X\in L(Y)\} \tag{4}$$

where $F_I(y,x)=D_I(y,x)^{-1}$.

In this paper, we seek to place additional structure on the production technology. In particular, we assume that production is homothetic.

Definition 2. A production function $\phi(X)$ is homothetic if

$$Y=\phi(X)=F(g(X))$$

where $F():R_+\to R_+$ is monotonically increasing and $g(\lambda X)=\lambda g(X)$, i.e., $g()$ is positive homogeneous of degree one and continuously differentiable (see Shephard, 1970). $g()$ is denoted the core function.

From the definition, we see that a homothetic production function can be represented as a production process whereby the input vector X can be aggregated into a one dimensional input index $g(X)$, i.e. output is determined from the level of aggregate input (see Färe & Lovell, 1988 for a more general result).

Proposition 1. *Assume a homothetic technology with one output. The distance function evaluated at* $(1,X)$ *is equal to aggregate input defined from the core function in the homothetic production function multiplied by a constant* k, *i.e.*

$$D_I(1,X)=k\times g(X),\quad k\in\mathbb{R}_+$$

Proof. Let $\phi(X)=F(g(X))$ with $F^{-1}=f$. We know that

[3] Recently, Olesen and Petersen (2013) suggested a method designed to provide a local (in the sense of a fixed input mix and a fixed output mix) estimation of lower and upper bounds on mpss. This suggested method requires "*two additional maintained hypotheses which imply that the DEA-frontier is consistent with smooth curves along rays in input and in output space that obey the Regular Ultra Passum (RUP) law, i.e. monotonically decreasing scale elasticities*", (abstract). The purpose of the approach suggested in this paper is different. We argue that the convex hull estimator is not consistent with standard microeconomic production theory that posits an S-shape for the production frontier, i.e. for production technologies that obey the Regular Ultra Passum Law.

[4] Frisch (1965, p. 89) discusses the economic theory of increasing marginal productivity that exists in a first stage of production at low levels of the variable input. In standard Principles of Microeconomics courses, production is illustrated with an S-shaped total product curve. The initial stage of increasing marginal returns is usually attributed to increases in specialization and division of labor. For example, Parkin (2014) argues that most production processes exhibit increasing marginal returns but eventually all production processes exhibit diminishing marginal returns.

[5] One of the referees has brought to our attention that a related paper entitled "Modeling Non-convex Production Frontiers: An Application to the Manufacturing Sector of China" by Sung-ko Li was presented at the North American Productivity Workshop VI in Houston in 2010. The paper is apparently unpublished. With one input and one output it seems to use an FDH approach below and a BCC approach above mpss.

[6] See Ruggiero (1996) and Podinovski (2005).