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Discrete Malliavin calculus and computations of greeks in the binomial tree

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ABSTRACT

This paper proposes new methods for computation of *greeks* using the binomial tree and the discrete Malliavin calculus. In the last decade, the Malliavin calculus has come to be considered as one of the main tools in financial mathematics. It is particularly important in the computation of greeks using Monte Carlo simulations. In previous studies, greeks were usually represented by expectation formulas that are derived from the Malliavin calculus and these expectations are computed using Monte Carlo simulations. On the other hand, the binomial tree approach can also be used to compute these expectations. In this article, we employ the *discrete Malliavin calculus* to obtain expectation formulas for greeks by the binomial tree method. All the results are obtained in an elementary manner.

1. Introduction

In the last decade, the Malliavin calculus has come to be considered as one of the main tools in financial mathematics. It is particularly important for computations of greeks. However, the Malliavin calculus appears to be a very difficult concept to understand for most practitioners. Because of these reasons, we employ the discrete Malliavin calculus on symmetric Bernoulli random walks. This approach enables us to compute *discrete Malliavin greeks* using only elementary mathematics.

Greeks are quantities that represent the sensitivities of the price of derivative securities with respect to changes in the underlying asset price or parameters. They are defined in terms of the derivative of the option price with respect to parameters. One such greek, *Delta*, measures the sensitivity of the option price with respect to changes in the asset price. It is defined by the first derivative of the option value function with respect to the asset price. Another greek, *Gamma*, measures the sensitivity in the delta with respect to changes in the underlying asset price. It is defined by the second derivative of the option value function with respect to the asset price. *Vega* measures the sensitivity of the option price with respect to changes in the volatility level, σ. It is defined by the first derivative of the option value function with respect to the volatility level. These sensitivities are often important for risk management. In mathematical finance, sensitivities for options with a complex pay-off structure are often considered. Fournié et al. (1999) employed the Malliavin calculus to compute greeks and overcome these problems. For further discussion, refer to Benhamou (2003) and Kohatsu-Higa and Montero (2003) among others. In addition, refer Nualart (2006); Nunno et al. (2008), and Privault (2009) for discussions on the Malliavin calculus. Greeks are derived by combinations of the Malliavin calculus and Monte Carlo simulations in most previous studies. On the other hand, another important tool for evaluating greeks for contingent claims is the binomial tree approach introduced by Cox et al. (1979). Although the naive finite difference approach is the most intuitive way to determine the sensitivity for the options value function, it is not a robust approach and thus not recommended. Pelsser and Vorst (1994) proposed a simple calculation for computing delta and gamma in the binomial tree approach. Also, refer to Hull (2008) for discussion on this topic. Although computational methods for delta and gamma have been investigated in these past studies, the computational method for the vega, for instance, is not clear if the simple finite difference approach is not used. There is an alternative method for computing greeks in the binomial tree approach. Rozario (2004) derived discrete Malliavin greeks in the binomial tree model by discretizing (continuous) Malliavin greeks. The effectiveness of this method is discussed in that article. However, an even more direct approach can be employed. Discrete Malliavin greeks can be obtained by the binomial tree approach. This approach is more direct because one does not have to consider the continuous time model. It directly leads expectation formulas for discrete Malliavin greeks in binomial tree settings. Interestingly, all the discussions on derivations of greeks with this approach are very similar to discussions in the continuous time model, when the discrete Malliavin derivative and the discrete Skorohod integrals are defined.

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The remainder of this paper is organized as follows. We introduce the discrete Malliavin calculus and the basic properties for the discrete Malliavin derivative and the discrete Skorohod integrals are also discussed. Leitz-Martini (2000) first introduced the discrete Malliavin calculus to obtain the discrete Ocone and Clark formulas. The discrete Malliavin calculus of Leitz-Martini (2000) was extended to more general cases by Privault (2008, 2009) for other applications. Leitz-Martini (2000) defined the discrete Skorohod integrals using discrete Wick products, and Privault (2008, 2009) defined them using the discrete Wiener–Chaos expansions. Although our definition on the discrete Skorohod integrals is more elementary than the definitions in these previous studies, they are equivalent each other. This is also discussed. The formulas for discrete Malliavin greeks after brief explanations on the binomial tree approach of Cox et al. (1979). Computational results are also presented in the last section.

2. Discrete Malliavin calculus

The discrete Malliavin derivative and the discrete Skorohod integrals are defined in this section. Further, we discuss the fundamental results of the discrete Malliavin calculus to evaluate greeks in the next section.

Let $\left\{\epsilon_i^{(p)}\right\}_{i=1,\ldots,N}$ be a sequence of identically independent random variables defined on a probability space $(\Omega,\mathcal{F},Q)$ with

$$Q\left[\epsilon_i^{(p)}=\sqrt{\Delta t}\right]=p,\quad Q\left[\epsilon_i^{(p)}=-\sqrt{\Delta t}\right]=1-p\quad(0<p<1).$$

As a special case, if a random variable $\epsilon_i^{(p)}$ is a symmetric Bernoulli random variable, *i.e.* $p = 1/2$, the random variable $\epsilon_i^{(1/2)}$ is denoted simply by ϵ_i. The time step Δt is fixed at $\Delta t = T/N$, where T is the horizontal time of our model. A filtration $\{\mathcal{F}_{i\Delta t}\}_{i=0,\ldots,N}$ is defined by $\mathcal{F}_0=\{\phi,\Omega\},\quad \mathcal{F}_{i\Delta t}=\sigma\left(\epsilon_1^{(p)},\ldots,\epsilon_i^{(p)}\right)$. We define that F as a random variable generated by $\left\{\epsilon_i^{(p)}\right\}_{i=1,\ldots,N}$, if there is a function $f(x_1,\ldots,x_N)$ such that $F=f\left(\epsilon_1^{(p)},\ldots,\epsilon_N^{(p)}\right)$. The random variable F is termed an $\mathcal{F}_{i\Delta t}$ measurable random variable, if there is a function $f(x_1,\ldots,x_i)$, such that $F=f\left(\epsilon_1^{(p)},\ldots,\epsilon_i^{(p)}\right)$. The symbols $\left(\epsilon_1^{(p)},\ldots,\epsilon_i^{(p)}\right)$ and $(\epsilon_1,\ldots,\epsilon_i)$ are denoted as $\left(e_i^{(p)}\right)$ and (e_i), respectively, for simplicity. A set of random variables $\{F_{i\Delta t}\}_{i=1,\ldots,N}$ defined on time $\{\Delta t, 2\Delta t,\ldots,N\Delta t\}$, which is generated by $\left\{\epsilon_i^{(p)}\right\}_{i=1,\ldots,N}$, is represented by a set of functions $\{f_i(x_1,\ldots,x_N)\}_{i=1,\ldots,N}$. It is expressed as

$$F_{i\Delta t}=\sum_{j=1}^{N} f_j\left(e_N^{(p)}\right)1_i(j),\tag{2.1}$$

where a function $1_i(j)$ is an indicator function, *i.e.*, $1_i(j) = 1\,(i = j)$ and $1_i(j) = 0\,(i\neq j)$. Then, a set of random variables $\{F_{i\Delta t}\}_{i=1,\ldots,N}$ is called a stochastic process generated by $\left\{\epsilon_i^{(p)}\right\}_{i=1,\ldots,N}$.

Example 2.1. A stochastic process $\left\{W_{i\Delta t}^{(p)}\right\}_{i=1,\ldots,N}$ is referred to as a random walk, if $W_{i\Delta t}^{(p)}$ is expressed as

$$W_{i\Delta t}^{(p)}=\sum_{j=1}^{N}\left(\epsilon_1^{(p)}+\cdots+\epsilon_i^{(p)}\right)1_i(j)=\epsilon_1^{(p)}+\cdots+\epsilon_i^{(p)}.$$

As a special case, if the upward probability p is fixed at $p = 1/2$, the stochastic process $W_{i\Delta t}^{(1/2)}$ is referred to as a symmetric random walk. This symmetric random walk is denoted by $W_{i\Delta t}^{(1/2)}=W_{i\Delta t}$ for abbreviation. The symmetric random walk is regarded as an approximation of the Brownian motion, if the time interval Δt is sufficiently small. The asymmetric random walk $\left\{W_{i\Delta t}^{(p)}\right\}_{i=1,\ldots,N}$ is also regarded as an approximation of the Brownian motion with a drift $\frac{2p-1}{\sqrt{\Delta t}}$.

A set of random variables $\{F_{i\Delta t,j\Delta t}\}_{i\neq j}^{i,j=1,\ldots,N}$ defined on the pairs of times $\{(i\Delta t,j\Delta t)\}_{i\neq j}^{i,j=1,\ldots,N}$ is represented by the functions $\{f_{i,j}(x_1,\ldots,x_N)\}_{i\neq j}^{i,j=1,\ldots,N}$. It is expressed as

$$F_{i\Delta t,j\Delta t}=\sum_{k,l=1,k\neq l}^{N} f_{i,j}\left(e_N^{(p)}\right)1_{i,j}(k,l),\tag{2.2}$$

where the function $1_{i,j}(k,l)$ is an indicator function, *i.e.*, $1_{i,j}(k,l) = 1$ ($i = k$ and $j = l$) and $1_{i,j}(k,l) = 0$ (*otherwise*). Then, a set of random variables $\{F_{i\Delta t,j\Delta t}\}_{i\neq j}^{i,j=1,\ldots,N}$ is called a generalized stochastic process generated by $\left\{\epsilon_i^{(p)}\right\}_{i=1,\ldots,N}$. We exploit the generalized stochastic process to compute gamma.

In this article, we only introduce the discrete Malliavin calculus on the symmetric random walk. However, one can also construct the discrete Malliavin calculus on the asymmetric random walk $W_{i\Delta t}^{(p)}$. Refer to Privault (2008, 2009) for discussion on that topic. Let us assume that there are two random variables $X(e_N)$ and $Y(e_N)$. The inner product on these random variables is defined by

$$\langle X,Y\rangle_{L^2}=E[X(e_N)Y(e_N)]=\frac{1}{2^N}\sum_{e_N\in\Omega}X(e_N)Y(e_N).$$

A set of random variables generated by $\{\epsilon_i\}_{i=1,\ldots,N}$ with a finite norm is denoted by $L^2(\Omega,Q)$. This set is called the *discrete Wiener space*.

Definition 2.1. The discrete Malliavin derivative for every random variable $F = f(e_N)\in L^2(\Omega,Q)$ is an operator from an element in $L^2(\Omega,Q)$ into a stochastic process; $\{D_{i\Delta t}F\}_{i=1,\ldots,N}$ is defined by

$$D_{i\Delta t}F=\sum_{j=1}^{N}\frac{f\left(e_N^{i+}\right)-f\left(e_N^{i-}\right)}{2\sqrt{\Delta t}}1_i(j)=\frac{f\left(e_N^{i+}\right)-f\left(e_N^{i-}\right)}{2\sqrt{\Delta t}},$$

where new symbols $\left(e_i^{k+}\right)$ and $\left(e_i^{k-}\right)$ are defined by

$$\left(e_i^{k\pm}\right)=\left(\epsilon_1,\ldots,\epsilon_{k-1},\pm\sqrt{\Delta t},\epsilon_{k+1},\ldots,\epsilon_i\right).$$

We will now demonstrate several examples of the discrete Malliavin derivative.

Example 2.2. A stochastic process $\left\{W_{i\Delta t}^{k\pm}\right\}_{i=1,\ldots,N}$ is a Bernoulli random walk with a deterministic epoch at time $k\Delta t$. In other words, the random variable ϵ_k is deterministic and is defined by

$$W_{i\Delta t}^{k\pm}=\epsilon_1+\cdots+\epsilon_{k-1}\pm\sqrt{\Delta t}+\epsilon_{k+1}+\cdots+\epsilon_i$$

if $k\leqslant i$ and $W_{i\Delta t}^{k\pm}=W_{i\Delta t}$ if $k > i$. Let us regard the random walk at time $i\Delta t$ as a random variable. Applying the discrete Malliavin derivative to $W_{i\Delta t}$ results in the following expression,

$$D_{k\Delta t}W_{i\Delta t}=\frac{W_{i\Delta t}^{k+}-W_{i\Delta t}^{k-}}{2\sqrt{\Delta t}}=1_{k\leqslant i}.$$

Example 2.3. Let $f(\cdot)$ be a differentiable function. Applying the discrete Malliavin derivative to a random variable $f(W_{i\Delta t})$ leads to

$$D_{k\Delta t}f(W_{i\Delta t})=\frac{f\left(W_{i\Delta t}^{k+}\right)-f\left(W_{i\Delta t}^{k-}\right)}{2\sqrt{\Delta t}}.\tag{2.3}$$

We consider three cases for (2.3).

- **Case 1.** $k > i$: The relation $W_{i\Delta t}^{k+}=W_{i\Delta t}^{k-}=W_{i\Delta t}$ is satisfied in this case. This relation leads to the immediate result $D_{k\Delta t}f(W_{i\Delta t}) = 0$.
- **Case 2.** $k\leqslant i$ **and** $\epsilon_k=\sqrt{\Delta t}$: The relations $W_{i\Delta t}^{k+}=W_{i\Delta t}$ and $W_{i\Delta t}^{k-}=W_{i\Delta t}-2\sqrt{\Delta t}$ are satisfied. If Δt is sufficiently small, this relation leads to the following result