European Journal of Operational Research 229 (2013) 453–461

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

A dynamic programming approach to constrained portfolios

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ARTICLE INFO

Article history:
Received 11 January 2012
Accepted 20 February 2013
Available online 13 March 2013

Keywords:
Finance
Markov processes
Consumption–investment problems
Utility maximization
Bellman equations

ABSTRACT

This paper studies constrained portfolio problems that may involve constraints on the probability or the expected size of a shortfall of wealth or consumption. Our first contribution is that we solve the problems by dynamic programming, which is in contrast to the existing literature that applies the martingale method. More precisely, we construct the non-separable value function by formalizing the optimal constrained terminal wealth to be a (conjectured) contingent claim on the optimal non-constrained terminal wealth. This is relevant by itself, but also opens up the opportunity to derive new solutions to constrained problems. As a second contribution, we thus derive new results for non-strict constraints on the shortfall of intermediate wealth and/or consumption.

1. Introduction

Classical dynamic portfolio optimization is concerned with solving non-constrained portfolio problems (see, e.g., Merton (1990)). In practice, a lot of realistic portfolio problems however involve constraints on wealth and consumption. This is because, for instance, financial institutions hold assets to support their obligations to contract holders and to satisfy other stakeholders. Particular examples of these financial institutions are pension funds that we use as a stylized example in this paper.

The objective of this paper is twofold: First, we make a methodological contribution by solving these constrained portfolio problems applying dynamic programming. The standard approach to dynamic portfolio optimization with constraints on wealth is the so-called martingale method. The martingale method was developed by Karatzas et al. (1987) and Cox and Huang (1989) as an alternative to dynamic programming. The method decomposes the dynamic optimization problem into a static optimization problem and a dynamic hedging problem where the latter one is usually involved.[3] On the contrary, dynamic programming gives easy access to the value function and the controls of the problem and thus plays an important role for solving stochastic control problems in finance. To the best of our knowledge, problems with *constraints on wealth* have not been analyzed by dynamic programming yet.[4] Our paper closes this gap and shows how to set up Hamilton–Jacobi-Bellman equations for problems with constraints on both consumption and wealth. We demonstrate how to solve the corresponding highly non-linear partial differential equations. From a stochastic control point of view, this is an important contribution by itself.

Furthermore, and this is our second main contribution, the dynamic programming approach also opens up the opportunity to solve constrained portfolio problems beyond the ones addressed in the literature so far. We are able to study new problems with constraints on intermediate consumption and/or wealth. This is possible because, in contrast to the martingale approach, dynamic programming does not introduce a static optimization problem that is decoupled from the corresponding dynamic hedging problem. For instance, we generalize the terminal utility problem considered by Basak and Shapiro (2001) to include intermediate consumption. Here, the fundamental ideas are adapted from Lakner and Nygren (2006), but since we allow for non-strict constraints the results are new. We introduce several ways to relax constraints on intermediate consumption. We formalize a problem where lump sum consumption at discrete time points is restricted by a value-at-risk (VaR) or an expected shortfall constraint. Another situation of special interest for pension asset managers is the case where there is no utility from (but still constraints on)

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[1] Holger Kraft gratefully acknowledges financial support by Deutsche Forschungsgemeinschaft (DFG).

[2] We thank the editor and anonymous referees for helpful comments and suggestions. All remaining errors are of course our own.

[3] Formally, this is because the martingale representation theorem only guarantees the existence of an optimal portfolio strategy, but does not provide guidance on how to construct a solution.

[4] Notice that wealth is a controlled process. Therefore, the problem is different from studying constraints on controls such as portfolio strategies. We will discuss this point in detail below.

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http://dx.doi.org/10.1016/j.ejor.2013.02.039

intermediate consumption. All these problems can be addressed with our approach.

Our results give guidance on how to allocate funds among assets that serve a dual purpose: On the one hand, the cash-flows from these assets go to stakeholders of an insurance company. This is described via a goal function that involves *utility of consumption* and/or *wealth*. On the other hand, the assets also protect future obligations (e.g. claims of policy holders) that are modeled via *constraints on* consumption and/or wealth. Problems with no constraints/utility on/from intermediate consumption (Basak and Shapiro (2001)) and with strict constraints on intermediate consumption and/or wealth (Lakner and Nygren (2006)) are included as special cases. There exists an extensive literature on various types of constrained portfolio problems. In general, one can distinguish between two different types of constraints: constraints on terminal wealth (''controlled process'') or constraints on portfolio strategies (''controls''). In this paper, we focus on problems with constraints on wealth and also add constraints on intermediate consumption.[5] We however abstract from constraints on portfolio strategies that were extensively studied in recent papers. Furthermore, the literature on consumption–portfolio optimization can also be distinguished w.r.t. the goal function of the problem. In particular, there are papers considering problems with utility maximization, whereas others study problems with classical criteria such as mean–variance maximization. Both problems are relevant, but have to be addressed by applying different methods.[6] Our paper concentrates on utility maximization. Finally, the work in this area can be distinguished w.r.t. whether martingale or dynamic programming methods are applied. As mentioned above, we establish a dynamic programming method to study consumption–portfolio problems with constraints on intermediate consumption and wealth. Both the method and some of the problems are new (e.g. non-strict constraints on consumption) and contribute to the existing literature. In the remainder of this section, we give a brief overview of this literature.

1.1. Constraints on terminal wealth

Grossman and Zhou (1996), Tepla (2001), and Korn (2005) study optimization problems with strict downward constraints on wealth.[7] Basak and Shapiro (2001) consider both relaxed downward constraints that can be violated with a certain probability (VaR constraints) and a constraint where the expected tail loss is restricted (expected shortfall constraint). Basak et al. (2006) and Boyle and Tan (2007) generalize the results for VaR constraints to the case where wealth must exceed a stochastic, but hedgeable, benchmark with a given probability. Korn and Wiese (2008) study the case where, essentially, the benchmark for the portfolio is a non-hedgeable insurance claim, but restrict to certain classes of portfolios with different types of homogeneity assumptions. All these papers use the martingale method[8] and, for instance, do not allow for constraints on intermediate consumption. *Constraints on intermediate consumption and wealth* are usually disregarded in portfolio insurance problems. An exception is Lakner and Nygren (2006) where not only the terminal wealth but also a continuous consumption rate is restricted downwards in a strict sense. As all other above-mentioned papers with constraints on wealth, Lakner and Nygren (2006) use the martingale approach. We distinguish ourselves by using dynamic programming and by allowing for non-strict constraints.

1.2. Constraints on portfolio strategies

Firstly, there are papers studying utility maximization problems with portfolio constraints. The classical reference is Cvitanic and Karatzas (1992) who apply duality methods to solve problems with cone constraints.[9] These papers disregard constraints on terminal wealth or intermediate consumption. Furthermore, there is an extensive body of research on the classical mean–variance problem that was originally developed for a static setting, but can be studied in a continuous-time dynamic setting as shown by Zhou and Li (2000). This problem can be combined with constraints on portfolio weights. Typically, such constraints are non-convex and computational methods have to be applied. Anagnostopoulos and Mamanis, 2008 and Branke et al., 2009 use evolutionary algorithms to search for optimal constrained portfolios in a mean–variance framework. Zhu et al. (2011) apply the particle swarm optimization approach to mean–variance portfolio optimization. Crama and Schyns (2003) solve constrained mean–variance problems by means of simulated annealing. These papers use so-called heuristic optimization methods in order to circumvent the challenges of non-convexity. Special cases of borrowing constraints have been solved by dynamic programming methods.[10] Besides, Emmer et al. (2001) show that portfolio-insurance-like strategies arise under a quadratic criterion. Osorio et al. (2008) show that a different type of constraints is relevant if mean–variance optimization of post-tax wealth in non-linear tax regimes is analyzed. We distinguish ourselves by working with convex constraints on wealth and consumption rather than portfolio strategies, by working with utility optimization rather than mean–variance optimization, and by working with dynamic programming.

The outline of the paper is as follows: Section 2 presents a general one-period problem and derives a sufficient condition for presenting the solution to an involved (constrained) investment problem as a contingent claim on the solution to a simple (unconstrained) investment problem. Section 3 exemplifies our results from Section 2 and derives the optimal portfolios for a simple linear case, a VaR constraint, and an expected shortfall constraint, respectively. Sections 4 and 5 generalize to intermediate consumption with constraints and to intermediate constraints on wealth. Some proofs can be found in the appendix.[11]

2. The portfolio problem and its value function

In this section, we relate the solution to an unconstrained portfolio problem to the solution of an involved constrained portfolio problem. We study the decisions of an investor (asset manager) operating in a standard Black–Scholes financial market with two assets, a bond (B) and a stock (S) the dynamics of which are given by

$$dB_t = rB_t dt, \quad B_0 = 1, \quad dS_t = S_t(\alpha dt + \sigma dW_t), \quad S_0 = s_0 > 0,$$

where r, α and σ are constants. The proportion of assets held in stocks is denoted by π such that the dynamics of the investor's wealth read

$$dX_t = (r + \pi_t(\alpha - r))X_t\,dt + \pi_t \sigma X_t\,dW_t,$$

[5] Depending on whether consumption is modeled as lump sum payments or as a continuous stream it can be interpreted as part of the goal function or as control. In this paper, we model consumption as lump sum payments, which is the more realistic case for insurance companies.

[6] For instance, it has been realized that continuous-time mean–variance problems are time-inconsistent, which is in contrast to utility-maximization problem, see Basak and Chabakauri (2010).

[7] See also Jensen and Sørensen (2001) for an application relevant for the above-mentioned pension fund managers.

[8] The only exception is Korn and Wiese (2008), but they face a different type of optimization problem.

[9] This is a generalization of the martingale method. See, e.g., Cvitanic and Zapatero (2004, Section 4.4), for further details and additional references.

[10] See, e.g., Fua et al. (2010) who allow for non-equal borrowing and lending interest rates.

[11] Longer versions of these proofs are available from the authors upon request.