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Decision Support

Multiattribute preference models with reference points ☆

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ABSTRACT

In the context of multiple attribute decision making, preference models making use of reference points in an ordinal way have recently been introduced in the literature. This text proposes an axiomatic analysis of such models, with a particular emphasis on the case in which there is only one reference point. Our analysis uses a general conjoint measurement model resting on the study of traces induced on attributes by the preference relation and using conditions guaranteeing that these traces are complete. Models using reference points are shown to be a particular case of this general model. The number of reference points is linked to the number of equivalence classes distinguished by the traces. When there is only one reference point, the induced traces are quite rough, distinguishing at most two distinct equivalence classes. We study the relation between the model using a single reference point and other preference models proposed in the literature, most notably models based on concordance and models based on a discrete Sugeno integral.

1. Introduction

In a series of papers, Rolland [26–30] (see also [25] in the related context of decision making under uncertainty) has suggested to use reference points[1] in an ordinal way to build preference models for multiattributed alternatives. This idea can be traced back to Fargier and Perny [15] and Dubois et al. [14, p. 247]. In these models, the preference between alternatives x and y rests on a comparison in terms of "importance" of the sets of attributes for which x and y are above the reference points. Rolland has analyzed the interest of such models and has proposed axioms that could characterize them. Most of his axiomatic analysis supposes that the reference points are known beforehand.[2] Including reference points in the primitives of the model is a strong hypothesis and raises observational questions. Moreover, he invokes conditions that seem to be quite specific to models using reference points. It is therefore not easy to use them in order to compare these models with other ones that have been proposed and characterized in the literature.

The aim of this text is to propose an axiomatic analysis of preference models with reference points using the traditional primitives of conjoint measurement, i.e., a preference relation on the set of alternatives. Our analysis uses a general conjoint measurement model resting on the study of traces induced on attributes by the preference relation and using conditions guaranteeing that these traces are complete [7]. We show that preference models with reference points are a particular case of this general model. This will allow us to characterize preference models with reference points using conditions that will facilitate their comparison with other preference models proposed in the literature.

We put a special emphasis on preference models that use a single reference point. On each attribute, these models induce traces that are quite rough, distinguishing at most two distinct equivalence classes. Our characterization of these models allows us to compare them with other types of preference models introduced in the literature. In particular, we will show that they are a particular case of models based on a discrete Sugeno integral and study their relations with models based on the notion of concordance.

Our general strategy will be similar to the one used in Bouyssou and Pirlot [9,10] to analyze models based on the notion of concordance (see also [5,11,19]). They have shown that such models could be seen as particular cases of the general conjoint measurement models developed in Bouyssou and Pirlot [4,6] that generate

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[1] The notion of "reference point" is unfortunately used in the literature with many different meanings. The interpretation of the reference points in the models studied in this paper is discussed below. These reference points have little to do with the reference point used in prospect theory to distinguish gains from losses [22,35] or from the reference points used as a crucial element in the framing of decisions [34].

[2] For exceptions, see Rolland [26, Th. 3] and Rolland [29, Section 2.1.2].

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complete traces on differences between levels in which these traces are "rough", i.e., only distinguishing a limited number of equivalence classes. We show here that models using reference points are a particular case of models inducing complete traces on levels developed in Bouyssou and Pirlot [7] in which these traces are "rough" (for a general overview of preference models based on different kinds of traces, we refer to [8]).

The paper is organized as follows. Section 2 introduces our notation and setting. Section 3 formalizes and illustrates preference models using a single reference point. Section 4 recalls the main ingredients of the general conjoint measurement models introduced in Bouyssou and Pirlot [7]. Section 5 characterizes preference models using a single reference point. Section 6 studies the links between preference models using a single reference point and other preference models introduced in the literature. Section 7 is devoted to the study of preference models using a single reference point that are weak orders. It also outlines an elicitation technique of the parameters of the model. Section 8 extends our results to preference models using several reference points. A final section discusses our findings. For space reasons and with apologies to the reader, most proofs are relegated to the supplementary material to this paper.

2. Background

2.1. Binary relations

A binary relation $\mathcal{K}$ on a set A is a subset of $A \times A$. We often write $a\,\mathcal{K}\,b$ instead of $(a, b) \in \mathcal{K}$. We define the symmetric and asymmetric parts of $\mathcal{K}$ as is usual.

An *equivalence* is a reflexive ($a\,\mathcal{K}\,a$), symmetric ($a\,\mathcal{K}\,b \Rightarrow b\,\mathcal{K}\,a$) and transitive ($[a\,\mathcal{K}\,b$ and $b\,\mathcal{K}\,c] \Rightarrow a\,\mathcal{K}\,c$) binary relation on A. An equivalence relation partitions A into equivalence classes. The set of equivalence classes induced by the equivalence $\mathcal{K}$ is denoted by $A/\mathcal{K}$.

A *weak order* is a complete ($a\,\mathcal{K}\,b$ or $b\,\mathcal{K}\,a$) and transitive binary relation. When $\mathcal{K}$ is a weak order on A, it is clear that the symmetric part of $\mathcal{K}$ is an equivalence. We often abuse terminology and speak of equivalence classes of the weak order $\mathcal{K}$ instead of the equivalence classes of its symmetric part. In this case, we also speak of the first, second, ..., last equivalence class of $\mathcal{K}$.

A *semiorder* is a reflexive ($a\,\mathcal{K}\,a$), Ferrers ($[a\,\mathcal{K}\,b$ and $c\,\mathcal{K}\,d]$ imply $[a\,\mathcal{K}\,d$ or $c\,\mathcal{K}\,b]$) and semitransitive ($[a\,\mathcal{K}\,b$ and $b\,\mathcal{K}\,c]$ imply $[a\,\mathcal{K}\,d$ or $d\,\mathcal{K}\,c]$) binary relation. If $\mathcal{K}$ is a semiorder, it is well known (see, e.g., [1, pp. 208 and 224]) that the relation $\mathcal{K}^{\circ}$ defined letting, for all $a, b, c \in A$,

$$a\,\mathcal{K}^{\circ}b \iff [[b\,\mathcal{K}\,c \Rightarrow a\,\mathcal{K}\,c] \text{ and } [c\,\mathcal{K}\,a \Rightarrow c\,\mathcal{K}\,b]],$$

is a weak order.

2.2. Notation

In this paper, $\succsim$ will always denote a binary relation on a set $X = \prod_{i=1}^{n} X_i$ with $n \geqslant 2$. Elements of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes and $\succsim$ as an "at least as good as" relation between these alternatives. We denote by $\succ$ (resp. $\sim$) the asymmetric (resp. symmetric) part of $\succsim$. A similar convention holds when $\succsim$ is starred, superscripted and/or subscripted.

For any nonempty subset J of the set of attributes N, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \in N \setminus J} X_i$). When $x, y \in X$, with customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. We sometimes omit braces around sets. For instance, when $J = \{i\}$ we write X_{-i} and (x_i, y_{-i}).

We say that attribute $i \in N$ is *influential* (for $\succsim$) if there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, a_{-i}) \not\succsim (w_i, b_{-i})$ and *degenerate* otherwise. A degenerate attribute has no influence whatsoever on the comparison of the elements of X and may be suppressed from N. As in Bouyssou and Pirlot [9], in order to avoid unnecessary minor complications, we suppose henceforth that *all attributes in N are influential.*

Let $J \subset N$ be a proper nonempty subset of attributes. We say that $\succsim$ is *independent* (see, e.g., [36, p. 30]) for J if, for all $x_J, y_J \in X_J$,

$$(x_J, z_{-J}) \succsim (y_J, z_{-J}), \quad \text{for some } z_{-J} \in X_{-J} \Rightarrow (x_J, w_{-J}) \succsim (y_J, w_{-J}),$$
$$\text{for all } w_{-J} \in X_{-J}.$$

If $\succsim$ is independent for all proper nonempty subsets of N, we say that $\succsim$ is *independent*. It is clear that $\succsim$ is independent iff $\succsim$ is independent for $N \setminus \{i\}$, for all $i \in N$.

A *capacity* on N is a real valued function μ on 2^N such that, for all $A, B \in 2^N$, $A \supseteq B \Rightarrow \mu(A) \geqslant \mu(B)$. The capacity μ on N is *normalized* if, furthermore, $\mu(\varnothing) = 0$ and $\mu(N) = 1$. All capacities used in this text will be normalized.

The *Möbius inverse* of a capacity is the real valued function m on 2^N such that, for all $S \subseteq N$, $m(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mu(T)$ (see, e.g., [13]). A capacity is said to be k-additive [18] if its Möbius inverse is null for all subsets containing $k + 1$ elements or more. Capacities that are 2-additive are known to be of manageable complexity, whereas already allowing much flexibility w.r.t. additive capacities, (i.e., 1-additive capacities, see [18,23]).

3. Preference models with a single reference point

3.1. Motivation

The model that we study was introduced by Rolland [26–30]. It has close connections with ELECTRE TRI [32, Chap. 6]. Remember that ELECTRE TRI is a technique used to assign alternatives to ordered categories. Suppose that there are only two categories: $\overline{\mathcal{A}}$ and $\overline{\mathcal{U}}$, $\overline{\mathcal{A}}$ being the best category. The limit between these two categories is indicated by a profile p that is at the same time the lower limit of $\overline{\mathcal{A}}$ and the upper limit of $\overline{\mathcal{U}}$. In the pessimistic version of ELECTRE TRI, an alternative $x \in X$ belongs to category $\overline{\mathcal{A}}$ iff this alternative is declared at least as good as p. The central originality of ELECTRE TRI lies in the definition of this "at least as good as" relation that is based on the notions of concordance and non-discordance. Ignoring here the non-discordance condition, an alternative $x \in X$ is "at least as good as" the profile p if a "sufficient majority" of attributes support this assertion. When preference and indifference thresholds are equal, this is done as follows. A semiorder T_i is defined on each attribute. The set of attributes supporting the proposition that $x \in X$ is at least as good as p is simply $\mathcal{T}(x) = \{i \in N : x_i T_i p_i\}$. A positive weight w_i is assigned to each attribute. These weights are supposed to be normalized so that $\sum_{i=1}^{n} w_i = 1$. The test for deciding whether the subset of attributes $\mathcal{T}(x)$ is "sufficiently important" is done comparing $\sum_{i \in \mathcal{T}(x)} w_i$ to a majority threshold $\lambda \in [0.5, 1]$. We have:

$$x \in \overline{\mathcal{A}} \iff \sum_{i \in \mathcal{T}(x)} w_i \geqslant \lambda.$$

Ordered partitions $\langle \overline{\mathcal{A}}, \overline{\mathcal{U}} \rangle$ of this type have been studied and characterized in Bouyssou and Marchant [2]. For the sequel, it will be useful to note that the concordance condition for testing if x is "at least as good as" p only distinguishes two kind of attributes: the ones for which $x_i\, T_i\, p_i$ and the ones for which this is not true. It does not make further distinctions among the attributes and, in particular, does not make use of the preference difference between x_i and p_i. Hence, the assignment of an alternative mainly rests on "ordinal considerations".