European Journal of Operational Research 229 (2013) 487–495

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Consistent modeling of risk averse behavior with spectral risk measures

Hans Peter Wächter, Thomas Mazzoni *

Department of Economics and Finance, University of Greifswald, Greifswald, Germany

ARTICLE INFO

Article history:
Received 7 February 2012
Accepted 1 March 2013
Available online 14 March 2013

Keywords:
Risk management
Utility theory
Spectral risk measures
Decision theory
Coherent risk measure

ABSTRACT

This paper clarifies the relation between decisions of a risk-averse decision maker, based on expected utility theory on the one hand, and spectral risk measures on the other.

We first demonstrate that recent approaches to this problem generally do not provide strongly consistent results, i.e. they fail to induce identical preference orders simultaneously with both concepts. Then we detail the relation between risk-averse decisions under the dual theory of choice and spectral risk measures. This relation is identified as the fundamental reason why it is not in general possible to establish a simple one-to-one mapping between expected utility theory and spectral risk measures.

We are nonetheless able to use spectral risk measures to model decisions obtained using expected utility theory. Interestingly, this implies that a given utility function corresponds to a whole family of risk spectra.

1. Introduction

The introduction of the concept of coherent risk measures in the financial literature (Artzner et al., 1997; Artzner et al., 1999) was a milestone in constructing risk measures, which are feasible both from an academic and a practitioners point of view, to overcome the well known shortcomings of Value-at-Risk (Jorion, 2006; Vozian, 2010; Krause, 2003). Among the possible classes of coherent risk measures, the class of spectral risk measures introduced by Acerbi (Acerbi, 2002; Acerbi, 2004), whose most prominent member is the Expected Shortfall (Tasche, 2002; Acerbi and Tasche, 2002; Rockafellar and Uryasev, 2002), has attracted considerable attention in the financial literature. Its general properties (Acerbi, 2002; Acerbi, 2004; Acerbi, 2007; Cherny, 2006; Dowd et al., 2008), estimation procedures (Acerbi, 2002; Giannopoulos and Tunaru, 2005; Cotter and Dowd, 2006; Cotter and Dowd, 2007a; Cotter and Dowd, 2007b) and its versatility in portfolio optimization (Rockafellar and Uryasev, 2000; Acerbi and Simonetti, in press; Adam et al., 2008; Strepparava, 2009; Deng et al., 2009; Brandtner, in press) have been studied extensively. In the context of portfolio optimization, the question of how to incorporate spectral risk measures into the framework of established theories of choice, is a particularly interesting and important one. This paper focuses on the problem of modeling rational behavior of a risk-averse decision maker, using spectral risk measures. For the dual theory of choice (Yaari, 1986; Yaari, 1987), this was already accomplished by Wang (1996), Wang (2000), Denuit et al. (2006), Gzyl and Mayoral (2006), using the equivalence of spectral risk measures and distortion risk measures. Recent research (Sriboonchitta et al., 2010; Tao et al., 2009; Dowd et al., 2008) tried to construct a similar connection between the classical theory of expected utility (von Neumann and Morgenstern, 1947; Föllmer and Schied, 2002) and spectral risk measures. Different approaches to this problem were proposed, but failed so far to provide a consistent theoretical framework.

This papers provides a conclusive answer to that question by elaborating a scheme, to consistently model the behavior of an expected-utility-maximizer within the class of spectral risk measures. Furthermore, the earlier approaches (Sriboonchitta et al., 2010; Tao et al., 2009; Dowd et al., 2008) and the connection to the dual theory of choice are discussed to provide a complete and self-contained picture. The remainder of this paper is organized as follows: Section 2 introduces spectral risk measures and their general relation to decision theories. Section 3 provides a brief review of recent attempts to establish a link between spectral risk measures and expected utility theory. Section 4 details the process and the implications of modeling risk averse behavior with spectral risk measures under the dual theory of choice. In Section 5, we provide a method for linking spectral risk measures and expected utility theory with the help of an auxiliary probability measure. This method is a close relative to the idea of changing probability measures in derivative pricing. We extend this approach to a general procedure in Section 6, and provide sufficient conditions for the link to be well defined and unique. We summarize and discuss our findings in Section 7.

2. General theory

Suppose $L^0(\Omega, \mathcal{F}, P)$ to be the space of all measurable, real-valued functions (i.e. random variables) on some probability space

* Corresponding author. Tel.: +49 3834 862498.
E-mail address: Thomas.Mazzoni@uni-greifswald.de (T. Mazzoni).

0377-2217/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.ejor.2013.03.001

$(\Omega, \mathcal{F}, P)$. Further, suppose the profit and loss (P&L) of a financial position to be determined at some future time T by the state of the world at that time, and to be fully described by some random variable $X \in L^0(\Omega, \mathcal{F}, P)$ or its cumulative distribution function (P&L-distribution) $F_X(x)$ today. In the following $F_X(x)$ is assumed to be monotonically increasing such that its quantile function $q_X(p) := F_X^{-1}(p)$ exists.

The functionals $\rho : L^0(\Omega, \mathcal{F}, P) \to \mathbb{R}$ are denoted as risk measures for the P&L-distributions, their subset

$$\rho_\varphi(X) := -\int_0^1 \varphi(p) F_X^{-1}(p) dp = -\int_0^1 \varphi(p) q_X(p) dp \tag{1}$$

of functionals being defined by any function $\varphi : [0,1] \to \mathbb{R}$, with the properties

$$\varphi(p) \geqslant 0 \quad \text{(positivity)}, \tag{2a}$$

$$\int_0^1 \varphi(p) dp = 1 \quad \text{(normalization), and} \tag{2b}$$

$$\varphi(p_1) \geqslant \varphi(p_2) \quad \text{(monotonicity), for any } p_1 \leqslant p_2 \tag{2c}$$

defines the set of spectral risk measures (cf. Acerbi, 2002; Acerbi, 2004; Tasche, 2002). The weighting function φ is also called the risk spectrum.

Now consider financial positions X and Y with P&L-distributions as introduced above, and further assume a decision maker with preference relations described by a functional $U : L^0(\Omega, \mathcal{F}, P) \to \mathbb{R}$ in the sense that:

$$X \text{ is preferred over } Y \iff U(X) > U(Y) \tag{3}$$

holds for arbitrary X, Y (cf. Puppe, 1991; Föllmer and Schied, 2002). A given theory of choice, describing a decision maker by some representation U, as introduced above, is now said to be consistently related to spectral risk measures ρ_φ, if U and φ can be mapped on each other such that

$$\rho_\varphi(X) \leqslant \rho_\varphi(Y) \iff U(X) \geqslant U(Y) \tag{4}$$

holds for all $X, Y \in L^0(\Omega, \mathcal{F}, P)$. This relationship implies that the rational behavior of a decision maker, described by the theory of choice at hand, can be modeled by spectral risk measures. In particular, the key concept of risk aversion can then be operationalized consistently within the framework of spectral risk measures.

At this point one might ask, why should one be interested in modeling rational behavior with spectral risk measures? The answer is strongly related to a critical property of expected utility theory, namely risk preferences are strictly associated to the utility function, independent of the respective P&L-distribution. On the other hand, risk is a concept closely related to uncertainty, and hence to the P&L-distribution. Thus, modeling rational behavior with spectral risk measures is the attempt to map the pure risk preference of a decision maker. If there is a unique connection in the sense of (4) between SRMs and expected utility theory, it might be possible to separate risk and utility preferences after all.

3. Spectral risk measures and expected utility theory

Define the shorthand notation $\mathrm{E}_u[X] := \mathrm{E}[u(X)]$ to indicate the expectation value of an arbitrary utility function with respect to the P&L-distribution of X. Expected utility theory assumes a particular representation of the preference relation (3), namely

$$U(X) = \int_{-\infty}^{\infty} u(x) dF_X(x) = \mathrm{E}_u[X] \tag{5}$$

with a utility function u, mapping onto the real numbers (cf. Föllmer and Schied, 2002). To represent a risk averse decision maker, $u' > 0$ and $u'' \leqslant 0$ (concave u) are assumed, which allow for the definition of a simple local measure of risk aversion

$$r_{\mathrm{PA}} = -u''/u', \tag{6}$$

known as the *Pratt-Arrow*-coefficient (Pratt, 1964; Arrow, 1971).

Several recent research papers deal with a relationship between spectral measures of risk and expected utility theory. The works of Dowd et al. (2008) and Tao et al. (2009) consider particular types of utility functions and simply translate them into risk spectra of a corresponding functional form. Dowd et al. (2008) conclude correctly that this ad hoc assignment of a risk spectrum, with respect to a given utility function, can lead to inconsistent results. Unfortunately, they attribute these problems to the general properties of spectral risk measures. We will show that these inconsistencies arise because of an inappropriate construction of the link between the utility function and the risk spectrum and not for more fundamental reasons. An arbitrary choice of such functions and subsequent interpretation of their parameters cannot be expected to yield consistent results.

In a recent publication Sriboonchitta et al. (2010) develop a calculation scheme for the systematic construction of a risk spectrum φ from a given utility function u. They define the buying price $p_B(X)$ of a random future P&L X, according to expected utility theory, such that $\mathrm{E}_u(X - p_B) = 0$ holds. Using results from robust statistics (e.g. Huber, 1981), they subsequently derive a scheme to construct a risk spectrum φ from a given utility function u, such that

$$\rho_\varphi(X) = -p_B(X) \tag{7}$$

holds. This identification is the central hypothesis of Sriboonchitta et al. (2010). The procedure is exercised only in the trivial case of a linear utility function $u(x) = x$, in which case $\varphi(p) = 1$ follows immediately. This leads to the fully consistent result $\rho_\varphi(-X) = -\mathrm{E}[X] = -p_B$. The non-trivial case of the exponential utility function $u(x) = 1 - e^{-kx}$ with arbitrary $k > 0$ was considered too involved for a direct computation by the authors. We provide this calculation in Appendix A. Based on this result, a closer investigation of the central hypothesis (7) reveals inconsistencies on a very fundamental level.

Consider the exponential utility function $u(x)$ and the random variables $X_1 \sim U(a, b)$ and $X_2 \sim N(\mu, \sigma)$. The buying price $p_B(X_1)$ is

$$p_B(X_1) = -\ln\left[\frac{e^{-ka} - e^{-kb}}{b - a}\right]. \tag{8}$$

The buying price $p_B(X_2)$ can be calculated by solving

$$\begin{aligned}\mathrm{E}_u[X_2 - p_B] &= \int_{-\infty}^{\infty} \frac{1 - e^{-k(x - p_B)}}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= 1 - \exp\left[-k(\mu - p_B) + \frac{k^2\sigma^2}{2}\right] = 0,\end{aligned} \tag{9}$$

which yields

$$p_B(X_2) = \mu - \frac{k}{2}\sigma^2. \tag{10}$$

In order for hypothesis (7) to be correct, $-p_B(X_1)$ and $-p_B(X_2)$ must obey the spectral risk measures axioms (cf. Acerbi, 2004). However, $-p_B(X_1)$ obviously violates the axiom of positive homogeneity and $-p_B(X_2)$ violates the axiom of monotonicity. It follows that neither $-p_B(X_1)$ nor $-p_B(X_2)$ defines a spectral risk measure (neither are even coherent) and thus hypothesis (7) is violated.

Summarizing these results, neither the ad hoc construction of Dowd et al. (2008) and Tao et al. (2009), nor the robust method of Sriboonchitta et al. (2010) succeeded in establishing a consistent relationship between expected utility theory and spectral risk measures.