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Invited Review

Packing and covering with linear programming: A survey

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ABSTRACT

This paper considers the polyhedral results and the min–max results on packing and covering problems of the decade. Since the strong perfect graph theorem (published in 2006), the main such results are available for the packing problem, however there are still important polyhedral questions that remain open. For the covering problem, the main questions are still open, although there has been important progress. We survey some of the main results with emphasis on those where linear programming and graph theory come together. They mainly concern the covering of cycles or dicycles in graphs or signed graphs, either with vertices or edges; this includes the multicut and integral multiflow problems.

1. Introduction

Although linear programming plays an important role in the design of efficient heuristics and exact algorithms for the packing and covering problems, we restrict this survey to the polyhedral and min–max results. For algorithmic aspects, we refer to the survey by Caprara et al. [19] and, for practical applications, to Balas [4]. Actually, Conforti et al. [29] already presented a survey on polyhedral and min–max aspects of packing–covering. So here, we will focus on the main results of the decade, that is, that appeared after 2001. We also refer to the book by Cornuéjols [31], to the survey by Cornuéjols and Guenin [32], and to the book by Schrijver [72]. A survey on packing–covering more concerned with graph theory is [10]. Surveys more concerned with approximation are [71,40].

The linear programming approach for solving a combinatorial optimization problem starts by modeling it as an *integer program* $z_{IP} := \max\{c^T x : x \in P \cap \mathbb{Z}^n\}$ where $P = \{x \in \mathbb{R}^n : Ax \leqslant b,\ x \geqslant 0\}$, A, b and c being respectively a rational matrix with m rows and n columns, an m-vector and an n-vector. Then we consider its *linear relaxation* $z_{LP} := \max\{c^T x : x \in P\}$ as well as its *dual* $\psi_{LP} := \min\{y^T b : y \in D\}$ where $D = \{y \in \mathbb{R}^m : y^T A \geqslant c^T, y \geqslant 0\}$. Finally we obtain the *integer dual* $\psi_{IP} := \min\{y^T b : y \in D \cap \mathbb{Z}^m\}$ of z_{IP}. We will suppose throughout the paper that both integer linear programs admit feasible solutions. Then we obtain $z_{IP} \leqslant z_{LP} = \psi_{LP} \leqslant \psi_{IP}$ where $z_{LP} = \psi_{LP}$ follows from the duality theorem of linear programming.

Notice that we abuse notation and use the same symbol for a linear program and its optimal value. The equality $z_{IP} = z_{LP}$ is equivalent to the fact that z_{LP} admits an integer optimal solution. In this case, ψ_{LP} does not necessarily admit an integer optimal solution, which makes a crucial difference since z_{IP} may be easy to solve while ψ_{IP} is not. Of course, the same holds if we exchange the role of z_{LP} and ψ_{LP}.

If $z_{IP} < z_{LP}$, more inequalities can be added to P without changing the value z_{IP} whatever the objective function $c^T x$ was. Such inequalities are said to be *valid*. Since it is easier to determine z_{LP} than z_{IP}, this approach is interesting if we succeed in decreasing z_{LP} so that $z_{IP} = z_{LP}$. For instance, if A is the 3×3 matrix with 0's on the diagonal and 1's elsewhere, and b is the all-one 3-vector, then the inequality $1^T x \leqslant 1$ is valid. Furthermore, adding it to P, we obtain $z_{IP} = z_{LP}$ for all c (since the only fractional extreme point of P, namely $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, is cut off).

A polyhedral result concerns the equality $z_{IP} = z_{LP}$ for all c. Then z_{LP} always admits an integer optimal solution and P is called an *integer polyhedron* [41]. A min–max result concerns the equality $z_{IP} = \psi_{IP}$ for some or for all c. Actually, if $\psi_{LP} = \psi_{IP}$ for all integer vectors c, then $z_{IP} = z_{LP}$ for all c, and then P is said to be *totally dual integral* (TDI, for short). So TDIness implies integrality, but the converse is not necessarily true. If $P \cap \{l \leqslant x \leqslant u\}$ is TDI for all l and u, then P is said to be *box-TDI*.

From now on, we let $A := A(\mathcal{H})$ and $\mathcal{H} := \mathcal{H}(A)$ be respectively a 0–1 matrix and a hypergraph such that A is the hyperedge–vertex incidence matrix of $\mathcal{H}$ (that is, the vertex set $V(\mathcal{H})$ of $\mathcal{H}$ is in 1-to-1 correspondence with the set of columns of A and the hyperedge set $E(\mathcal{H})$ of $\mathcal{H}$ is in 1-to-1 correspondence with the set of rows of A where each row is the 0–1 characteristic vector of its corresponding

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hyperedge). A subset $T \subseteq V(\mathcal{H})$ is a *vertex cover of* $\mathcal{H}$ if each $e \in E(\mathcal{H})$ contains at least one element in T. A subset $S \subseteq V(\mathcal{H})$ is a *stable set of* $\mathcal{H}$ if no hyperedege contains more than one vertex in S. Throughout the paper, w denotes an n-vector which associates a weight to every vertex of $\mathcal{H}$.

The *packing problem* consists in finding a *maximum-weight stable set of* $\mathcal{H}$, that is determining an optimal solution of the integer program $\alpha_w(\mathcal{H})$, defined below together with its relaxation $\alpha^*_w(\mathcal{H}) = \rho^*_w(\mathcal{H})$ and its integer dual $\rho_w(\mathcal{H})$:

$$\alpha_w(\mathcal{H}) := \max\{w^T x : Ax \leqslant 1,\ x \geqslant 0,\ x \in \mathbb{Z}^{V(\mathcal{H})}\} \leqslant \max\{w^T x : Ax \leqslant 1,\ x \geqslant 0,\ x \in \mathbb{R}^{V(\mathcal{H})}\} := \alpha^*_w(\mathcal{H}) = \min\{1^T y : A^T y \geqslant w,\ y \geqslant 0,\ y \in \mathbb{R}^{E(\mathcal{H})}\} := \rho^*_w(\mathcal{H}) \leqslant \min\{1^T y : A^T y \geqslant w,\ y \geqslant 0,\ y \in \mathbb{Z}^{E(\mathcal{H})}\} := \rho_w(\mathcal{H})$$

The *covering problem* consists in finding a *minimum-weight vertex cover of* $\mathcal{H}$, that is determining $\tau_w(\mathcal{H})$ defined below together with its relaxation $\tau^*_w(\mathcal{H}) = \nu^*_w(\mathcal{H})$ and its integer dual $\nu_w(\mathcal{H})$:

$$\tau_w(\mathcal{H}) := \min\{w^T x : Ax \geqslant 1,\ x \geqslant 0,\ x \in \mathbb{Z}^{V(\mathcal{H})}\} \geqslant \min\{w^T x : Ax \geqslant 1,\ x \geqslant 0,\ x \in \mathbb{R}^{V(\mathcal{H})}\} := \tau^*_w(\mathcal{H}) = \max\{1^T y : A^T y \leqslant w,\ y \geqslant 0,\ y \in \mathbb{R}^{E(\mathcal{H})}\} := \nu^*_w(\mathcal{H}) \geqslant \max\{1^T y : A^T y \leqslant w,\ y \geqslant 0,\ y \in \mathbb{Z}^{E(\mathcal{H})}\} := \nu_w(\mathcal{H})$$

When $w = 1$, we simply write $\alpha(\mathcal{H}),\ \alpha^*(\mathcal{H})$ and so on. For the packing/covering problem we may assume, without loss of generality, that w is nonnegative. Moreover we may assume, without loss of generality, that $\mathcal{H}$ has no isolated vertices and that no hyperedge contains another hyperedge. Thus $E(\mathcal{H})$ suffices to define $\mathcal{H}$; in this case $\mathcal{H}$ is called a *clutter* and A a *clutter-matrix*.

Many well-known theorems in combinatorial optimization are min–max packing–covering relations. See for instance [32] for an accessible tutorial on integrality in linear programs modeling covering problems (see also the books [31,72]). Let us consider some examples. First, if $\mathcal{H} = G$ is a graph, then Kőnig's theorem states that $\tau(\mathcal{H}) = \nu(\mathcal{H})$ if G is bipartite. The weighted version of this result, known as Egerváry's theorem, states that $\tau_w(\mathcal{H}) = \nu_w(\mathcal{H})$ for all w if and only if G is bipartite. Galai's identities give $\alpha_w(\mathcal{H}) = \rho_w(\mathcal{H})$ for all w if and only if G is bipartite. If $\mathcal{H}$ is a hypergraph such that $V(\mathcal{H})$ is the edge set of some graph G, s and t are two distinct vertices of G, and $E(\mathcal{H})$ is the set of st-paths of G, then Menger's theorem states that $\tau(\mathcal{H}) = \nu(\mathcal{H})$. The weighted version of this result, that is $\tau_w(\mathcal{H}) = \nu_w(\mathcal{H})$ for all nonnegative integral w, is the max-flow/min-cut theorem. If $\mathcal{H}$ is a hypergraph such that $V(\mathcal{H})$ is the vertex set of a graph G and $E(\mathcal{H})$ is the set of maximal stable sets of G, then $\alpha_w(\mathcal{H}) = \rho_w(\mathcal{H})$ for all 0–1 valued w is equivalent to the fact that the clique number $\omega(H)$ is equal to the chromatic number $\chi(H)$ for every induced subgraph H of G, which means exactly that G is a perfect graph. It is important here to ask equality, not only for G, but for all its induced subgraphs as well, otherwise we maytake any graph G and add a big clique to obtain $\omega(G) = \chi(G)$. More generally, if $\tau(\mathcal{H}) = \nu(\mathcal{H})$ for a class of hypergraphs $\mathcal{H}$ closed under taking induced sub-hypergraphs, then $\tau_w(\mathcal{H}) = \nu_w(\mathcal{H})$ for any 0–1 valued w.

The min–max relation may only hold for a small class of hypergraphs although a polyhedral result exists for a large class with another formulation. A forest in a graph G can be seen as the complement of a vertex cover in a hypergraph $\mathcal{H}$, where $V(\mathcal{H})$ is the edge set of G and $E(\mathcal{H})$ is the set of cycles of G. Note that, for such a hypergraph $\mathcal{H}$, if G is the simple graph with four vertices and five edges, then $\tau(\mathcal{H}) = 2 > 3/2 = \tau^*(\mathcal{H})$. Edmonds' forest polytope theorem states that if we consider the linear program ψ_{LP} obtained by adding to $\tau^*_w(\mathcal{H})$ some valid inequalities, namely $\sum_{e \in E(U)} x_e \geqslant |E(U)| - |U| + 1$ for each nonempty $U \subseteq V(G)$, then one has $\tau_w(\mathcal{H}) = \psi_{LP}$ for all graphs G (and for all w). (The validity of the inequalities follows from the fact that a forest has at most $|U| - 1$ edges in the set $E(U)$ of edges induced by U.) A matching in a graph G can be seen as a stable set in a hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = E(G)$ and $E(\mathcal{H})$ the set of the stars of G. So Kőnig–Egerváry's theorem gives $\alpha_w(\mathcal{H}) = \rho_w(\mathcal{H})$ with G being bipartite. But if G is for instance a triangle, we have $\alpha(\mathcal{H}) = 1 < 3/2 = \alpha^*(\mathcal{H})$. Edmonds' matching polytope theorem states if we add to $\alpha^*_w(\mathcal{H})$ the valid inequalities $\sum_{e \in E(U)} x_e \leqslant \lfloor |U|/2 \rfloor$ for each set $U \subseteq V(G)$ containing an odd number of vertices, then we obtain a linear program z_{LP} satisfying $\alpha_w(\mathcal{H}) = z_{LP}$ for all graphs G and for all weights w. Actually, both linear systems, for forests and matchings, are TDI and even box-TDI for the forests.

Since 2006, thanks to the strong perfect graph theorem [26], the 0–1 matrices satisfying $\alpha_w(\mathcal{H}) = \alpha^*_w(\mathcal{H})$ or $\alpha_w(\mathcal{H}) = \rho_w(\mathcal{H})$ are well described, and can be recognized easily. This closes Berge's conjecture from 1960 (see Section 3.1). Concerning polyhedral results of the type $\alpha_w(\mathcal{H}) = z_{LP}$, there has been interesting progress for a subclass of claw-free graphs, namely quasi-line graphs [27,42] (see Section 3.2).

Concerning the covering problem, there is not even a conjecture for a characterization of 0–1 matrices such that $\tau_w(\mathcal{H}) = \tau^*_w(\mathcal{H})$ or $\tau_w(\mathcal{H}) = \nu_w(\mathcal{H})$. However, for binary hypergraphs (see Section 2.6), those with $\tau_w(\mathcal{H}) = \nu_w(\mathcal{H})$ have been characterized by Seymour in [77], and there is a conjecture of Seymour for those with $\tau_w(\mathcal{H}) = \tau^*_w(\mathcal{H})$ that is still open (since 1977), but has been partially solved in several ways, e.g. in 2002 by Cornuéjols and Guenin [33], see also [32,50,54]. A related result is the generalization of the four-color theorem to graphs containing no odd minor isomorphic to K_5 [58] (see Section 3.3). Concerning the hypergraphs without the binary property, there have been interesting results for the cycle–vertex and for the vertex–cycle incidence matrices in graphs or digraphs, and also for path/tree–edge incidence matrices [2,18,22,23,37–39,55]. A conjecture of Gallai from 1963 about covering vertices with dicycles and the maximum cardinality of a stable set in a strongly connected digraph has been solved in 2007 and there is a simple and very interesting proof based on total dual unimodularity [13,21,76].

Our survey is organized as follows. In Section 2, we recall the fundamental definitions and results that are needed to understand the new results of the decade. In Section 3, we survey the results concerning the packing problem. In Section 4, we present results concerning odd cycles and odd paths. Section 5 adresses multiflow problems. Section 6 deals with covering cycles in graphs. Finally, in Section 7 we present the results about dicycles in digraphs.

2. The general theory

2.1. Perfectness, idealness and Mengerianity

The most natural polyhedral and min–max questions for the packing–covering problem are related to the following properties:

- For the packing problem, A and $\mathcal{H}$ are called *perfect* if (a)–(c), which are surprisingly equivalent (see [31,72]), hold.
  (a) $\alpha_w(\mathcal{H}) = \alpha^*_w(\mathcal{H})$ for any 0–1 vector w,
  (b) $\alpha_w(\mathcal{H}) = \rho_w(\mathcal{H})$ for any integer vector w,
  (c) $E(\mathcal{H}) = \{$(maximal) cliques of$G\}$ for some perfect graph G.

- For the covering problem, we obtain the following (see [32]): properties (i)–(ii) are equivalent; furthermore, (iii) trivially implies (iv) which implies (i)–(ii) and trivially (v); finally (v) trivially implies (vi).
  (i) $\tau_w(\mathcal{H}) = \tau^*_w(\mathcal{H})$ for any vector w ($\mathcal{H}$ and A are called *ideal*),
  (ii) $\tau_w(\mathcal{H}) = \tau^*_w(\mathcal{H})$ for any 0–1–∞ vector w,