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Reliability and covariance estimation of weighted k-out-of-n multi-state systems

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ABSTRACT

In the literature of reliability engineering, reliability of the weighted k-out-of-n system can be calculated using component reliability based on the structure function. The calculation usually assumes that the true component reliability is completely known. However, this is not the case in practical applications. Instead, component reliability has to be estimated using empirical sample data. Uncertainty arises during this estimation process and propagates to the system level. This paper studies the propagation mechanism of estimation uncertainty through the universal generating function method. Equations of the complete solution including the unbiased system reliability estimator and the corresponding unbiased covariance estimator are derived. This is a unified approach. It can be applied to weighted k-out-of-n systems with multi-state components, to weighted k-out-of-n systems with binary components, and to simple series and parallel systems. It may also serve as building blocks to derive estimators of system reliability and uncertainty measures for more complicated systems.

1. Introduction

Reliability is widely recognized as a critical metric for system design, operation and maintenance. It is defined as the ability of a system to perform the required functions under stated conditions for a specified amount of time. System reliability is referred to in terms of a probability. Theoretically, it can be *directly* estimated based on collected data from reliability tests on a large amount of systems of the same type, or based on historical field data of such systems. This estimation approach would be impractical when the reliability test for such system is too expensive, or when a new type of system is designed without historical data. However, for many applications, the entire system can generally be decomposed into a collection of components that perform simpler functions each. The reliability estimations of these small components are less expensive from test data or more accessible through historical data than the entire system. Under these circumstances, system reliability can be *indirectly* estimated based on the estimated component reliability and the functional relationship between these components and the entire system.

According to sampling theory and estimation theory in statistical inference (Bernstein and Bernstein, 1999), the estimation of reliability at the component level based on historical or testing data is actually making inferences about characteristics of the entire population (the same type of component) based on characteristics of available samples. A complete solution to an estimation problem should not only include the estimate itself, but also a measure of uncertainty to describe how certain it is about the estimate. When the indirect method is used to estimate system reliability, the estimation uncertainty at *the component level* propagates to *the system level*.

Indirect system reliability estimation considering uncertainty has been examined in many publications since 1960s. Some examples in this area are (Mawaziny and Buehler, 1967; Lieberman and Ross, 1971; Winterbottom, 1974; Gertsbakh, 1982; Ushakov, 1994). Extended studies have been conducted by Coit (1997), Ramirez-Marquez and Jiang (2006a,b), and Jin and Coit (2008). These studies have made important contributions to deriving exact or approximate expressions of the system reliability and the associated uncertainty measures. The derivation explains how uncertainty in the component-level reliability estimation propagates to the system level. All these studies, however, are for traditional *binary systems* with series or parallel configurations. These reliability models only allow the components and the system to experience two states: operating or failed. More practical reliability models have introduced the concept of partial failures and multiple failure modes to provide finer insights into the degradation behavior of a component or a system. Systems based on these models are referred to as *multi-state systems* (Aven, 1993; Brunelle and Kapur, 1999; Lisnianski and Levitin, 2003; Jane and Laih, 2010). For binary systems, binomial testing data is generally used as the basis for indirect system reliability estimation. Commonly used uncertainty measures are variance and confidence intervals of the reliability

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Notation and abbreviations

UGF	universal generating function
k	required total weight of the weighted k-out-of-n system
n	number of components in the weighted k-out-of-n system
v, v'	virtual components
i	index of components, $i \in \{1,2,\ldots,n\}$ or $i \in \{v, v'\}$
s_i	total number of states of component i
$j, j_1, j_2, j_3, j_4, j_5, j_6$	indexes of states, $j, j_1, j_2, j_3, j_4, j_5, j_6 \in \{1,2,\ldots,s_i\}$
$w_{i,j}$	weight of component i when it is in state j
$\mathbf{w}_i$	weight vector of component i, $\mathbf{w}_i = [w_{i,1}, w_{i,2}, \ldots, w_{i,s_i}]^T$
m_i	total number of components of the ith type put in the multinomial experiment
$y_{i,j}$	total number of components of the ith type that end up in state j in the multinomial experiment
$\mathbf{y}_i$	result vector of the multinomial experiment, $\mathbf{y}_i = [y_{i,1}, y_{i,2}, \ldots, y_{i,s_i}]^T$
$p_{i,j}$	probability that component i would end up in state j
$\hat{p}_{i,j}$	unbiased estimator of $p_{i,j}$
$\mathbf{p}_i$	probability vector of component i, $\mathbf{p}_i = [p_{i,1}, p_{i,2}, \ldots, p_{i,s_i}]^T$
$\hat{\mathbf{p}}_i$	unbiased estimator of $\mathbf{p}_i$
σ_{i,j_1,j_2}	covariance between $\hat{p}_{i,j_1}$ and $\hat{p}_{i,j_2}$
$\hat{\sigma}_{i,j_1,j_2}$	unbiased estimator of σ_{i,j_1,j_2}
$\mathbf{\Sigma}_i$	covariance matrix of component i
$\hat{\mathbf{\Sigma}}_i$	unbiased estimator of $\mathbf{\Sigma}_i$
$r(k,n)$	reliability of the weighted k-out-of-n system
$\hat{r}(k,n)$	unbiased estimator of $r(k,n)$
$u_i(z)$	UGF of component i

estimate (Coit, 1997; Ramirez-Marquez and Jiang, 2006b). However, for multi-state systems, the estimation will be based on the multinomial testing data (Bernstein and Bernstein, 1999; Evans et al., 2000). The covariance matrix should be utilized as the uncertainty measure since it contains covariance between elements of a random vector of multi-state reliability values. It is the natural generalization to higher dimensions of the concept of the variance of a scalar-valued random variable reliability of binary systems.

A special case of the multi-state system is the weighted k-out-of-n system (Li and Zuo, 2008), which can find abundant applications in various industries requiring redundancy and fault-tolerant performance (Tian et al., 2008). The weighted k-out-of-n system and its reliability are defined as follows.

Definition 1. The weighted k-out-of-n system contains n mutually independent components. The ith component ($i \in \{1,2,\ldots,n\}$) has s_i different states. It is in state j ($j \in \{1,2,\ldots,s_i\}$) with a probability $p_{i,j}$. When it is in state j, it carries a weight $w_{i,j}$. According to different combinations of operational states of the components, there are in total $\prod_{i=1}^{n} s_i$ possible system states, each of which corresponds to a possible total weight of all components and a probability. For a specific value k, the reliability $r(k,n)$ of the weighted k-out-of-n system with multi-state components is the probability that the total weight of all components is greater than or equal to k.

Remark 1. The physical meaning of the weight $w_{i,j}$ could be some performance characteristic of the component such as its transmission capacity or computation speed.

Remark 2. Although the total number of the possible system states is $\prod_{i=1}^{n} s_i$ according to the component states combination, some of these system states are unique and some of them are duplicate and indistinguishable.

Remark 3. n is the number of the components in the system, it is a positive integer. k characterizes a specific performance level (or weight) the system must exceed. Due to their different attributes, it is possible that $k > n$.

This definition is highly compatible. The following systems can be derived from Definition 1.

Definition 2. The weighted k-out-of-n system in Definition 1 reduces to the weighted k-out-of-n system with *only* binary components (Wu and Chen, 1994) when each component has only two states (that is, $s_i = 2$ for all $i \in \{1,2,\ldots,n\}$).

Definition 3. The weighted k-out-of-n system with binary components in Definition 2 reduces to the traditional non-weighted k-out-of-n system with binary components (Barlow and Heidtmann, 1984) if each component has a weight 1 when it is in the working state and a weight 0 when it is in the failed state (that is, $w_{i,1} = 1$ and $w_{i,2} = 0$, for all $i \in \{1,2,\ldots,n\}$).

Definition 4. The non-weighted k-out-of-n system with binary components in Definition 3 reduces to the simple parallel system when $k = 1$.

Definition 5. The non-weighted k-out-of-n system with binary components in Definition 3 reduces to the simple series system when $k = n$.

Furthermore, Definition 1 is very flexible. With different k and n values, the system can provide different levels of redundancy for customer-oriented applications. However, the direct method of system reliability estimation would be less economical or convenient for each system with a different (k,n) combination, and it is highly desirable to perform the indirect estimation method and study the uncertainty propagation mechanism. The objective of this paper is to mathematically derive the equations of the complete solution to the system reliability estimation problem. More specifically, we propose an approach to calculate the unbiased system reliability estimation and the corresponding unbiased variance (or covariance) estimation based on operational data or testing data of components.

The rest of this paper is organized as follows. Section 2 describes the universal generating function method in reliability calculation for the weighted k-out-of-n multi-state system. Section 3 introduces the multinomial experiment on components and derives estimators of reliability and the uncertainty measure. Proofs of all the propositions are also provided in this section. Examples of hypothetical and real-world applications are provided in Section 4 to demonstrate the proposed estimation approach. Conclusions of this paper are provided in Section 5.

2. The UGF method

According to Li and Zuo (2008), if the true reliability of components are known for a given period of mission time, the reliability of the weighted k-out-of-n system can be calculated using the universal generating function (UGF) method. The UGF method was first developed by Ushakov (1986, 2000) and then greatly extended by Lisnianski and Levitin (2003), Levitin (2005), and other