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Continuous Optimization

Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems ☆

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ABSTRACT

In this paper, we present an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem and the set of fixed points of an infinite family of nonexpansive mappings and the set of a variational inclusion in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithm has strong convergence under some mild conditions imposed on algorithm parameters.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a nonlinear mapping and let $\varphi : C \to R$ be a function and Θ be a bifunction of $C \times C$ into R.

Now, we consider the following mixed equilibrium problem: find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Fx^*, y - x^* \rangle \geqslant 0, \quad \forall y \in C. \tag{1.1}$$

If $F = 0$, then the mixed equilibrium problem (1.1) becomes the following mixed equilibrium problem: find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geqslant 0, \quad \forall y \in C, \tag{1.2}$$

which was considered by Ceng and Yao [1].

If $\varphi = 0$, then the mixed equilibrium problem (1.1) becomes the following equilibrium problem: find $x^* \in C$ such that

$$\Theta(x^*, y) + \langle Fx^*, y - x^* \rangle \geqslant 0, \quad \forall y \in C, \tag{1.3}$$

which was studied by Takahashi and Takahashi [2].

If $\varphi = 0$ and $F = 0$, then the mixed equilibrium problem (1.1) becomes the following equilibrium problem: find $x^* \in C$ such that

$$\Theta(x^*, y) \geqslant 0, \quad \forall y \in C. \tag{1.4}$$

If $\Theta(x, y) = 0$ for all $x, y \in C$, the mixed equilibrium problem (1.1) becomes the following variational inequality problem: find $x^* \in C$ such that

$$\varphi(y) - \varphi(x^*) + \langle Fx^*, y - x^* \rangle \geqslant 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of the problems (1.1)–(1.5) are denoted by $EP(1)$–$EP(5)$, respectively.

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The mixed equilibrium problems include optimization problems, variational inequality problems and the equilibrium problems as special cases. Related works (see [3–5,31,32] and the references therein) and some methods have been proposed to solve the mixed equilibrium problems and the equilibrium problems (see, for instance, [6–13,30]).

In 1997, Combettes and Hirstoaga [14] first introduced an iterative method of finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, some related works have been extended by many authors (see, for example, Ceng and Yao [1], Takahashi and Takahashi [9], Yao et al. [12], Peng and Yao [15], Suzuki [28], Xu [29]).

Recall some kinds of nonlinear mappings as follows:

(1) A mapping $f : C \to C$ is called a ρ-contraction if there exists a constant $\rho \in [0,1)$ such that

$$\|f(x) - f(y)\| \leqslant \rho\|x - y\|, \quad \forall x, y \in C.$$

(2) A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

Denote the set of fixed points of T by $Fix(T)$.

(3) A mapping $B : C \to C$ is said to be β-inverse strongly monotone if there exist a constant $\beta > 0$ such that

$$\langle Bx - By, x - y\rangle \geqslant \beta\|Bx - By\|^2, \quad \forall x, y \in C.$$

(4) A mapping A is strongly positive on H if there exists a constant $\mu > 0$ such that

$$\langle Ax, x\rangle \geqslant \mu\|x\|^2, \quad \forall x \in H.$$

Let $B : H \to H$ be a single-valued nonlinear mapping and $R : H \to 2^H$ be a set-valued mapping.

Now, we concern the following variational inclusion problem: find a point $x \in H$ such that

$$\theta \in B(x) + R(x), \tag{1.6}$$

where θ is the zero vector in H.

The set of solutions of the problem (1.6) is denoted by $I(B,R)$. If $H = R^m$, then the problem (1.6) becomes the generalized equation introduced by Robinson [16]. If $B = 0$, then the problem (1.6) becomes the inclusion problem introduced by Rockafellar [17]. It is known that the problem (1.6) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory, etc. Also, various types of variational inclusions problems have been extended and generalized, for more details, refer to [18–24] and the references therein.

Inspired and motivated by the works in the literature, in this paper, we present a new iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem and the set of fixed points of a nonexpansive mapping and the set of a variational inclusion in a real Hilbert space. Furthermore, we prove that the proposed iterative algorithm is strongly convergent under some mild conditions imposed on algorithm parameters.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leqslant \|x - y\|, \quad \forall y \in C.$$

Such a mapping P_C is called the metric projection of H onto C.

We know that P_C is nonexpansive and, further, for any $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \iff \langle x - x^*, x^* - y\rangle \geqslant 0, \quad \forall y \in C.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geqslant 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if, for any $(x,f) \in H \times H$, $\langle x - y, f - g\rangle \geqslant 0$ for all $(y,g) \in G(T)$ implies $f \in Tx$.

Let a set-valued mapping $R : H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}(x), \quad \forall x \in H,$$

where λ is a positive number.

It is worth mentioning that the resolvent operator $J_{R,\lambda}$ is single-valued, nonexpansive and 1-inverse strongly monotone (see, for example, [25]) and a solution of the problem (1.6) is a fixed point of the operator $J_{R,\lambda}(I - \lambda B)$ for all $\lambda > 0$ (see, for instance, [26]).

Throughout this paper, we assume that a bifunction $\Theta : H \times H \to \mathbf{R}$ and a convex function $\varphi : H \to R$ satisfy the following conditions:

(H1) $\Theta(x,x) = 0$ for all $x \in H$;
(H2) Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \leqslant 0$ for all $x, y \in H$;
(H3) for any $y \in H$, $x \mapsto \Theta(x,y)$ is weakly upper semicontinuous;
(H4) for any $x \in H$, $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous;
(H5) For any $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset H$ and $y_x \in H$ such that, for any $z \in H \setminus D_x$,