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Continuous Optimization

Inverse optimization for linearly constrained convex separable programming problems ☆

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ABSTRACT

In this paper, we study inverse optimization for linearly constrained convex separable programming problems that have wide applications in industrial and managerial areas. For a given feasible point of a convex separable program, the inverse optimization is to determine whether the feasible point can be made optimal by adjusting the parameter values in the problem, and when the answer is positive, find the parameter values that have the smallest adjustments. A sufficient and necessary condition is given for a feasible point to be able to become optimal by adjusting parameter values. Inverse optimization formulations are presented with ℓ_1 and ℓ_2 norms. These inverse optimization problems are either linear programming when ℓ_1 norm is used in the formulation, or convex quadratic separable programming when ℓ_2 norm is used.

1. Introduction

Inverse optimization has been studied widely. Let $\min P(x,c)$, $x \in X$, be an optimization problem where X is the feasible region, and c is an parameter vector representing costs, capacities, weights, returns, etc. The general optimization problem (also called forward optimization) is to find an $x^* \in X$ such that the objective $P(x,c)$ is optimal at x^*. The inverse optimization can be described as follows. Assume that we know an $\hat{x} \in X$ and a vector c which is an estimate to the parameter values of the problem. We are interested in finding out whether there exist acceptable values of the parameter vector c, denoted by $\hat{c}$, which make $\hat{x}$ optimal to problem $\min P(x,\hat{c})$. If the answer is positive, find the vector $\hat{c}$ that differs from the vector c as little as possible.

Burton and Toint [10,11] are pioneers working in inverse optimization. In [10,11] they first investigated an inverse shortest paths problem. Since then, many different inverse optimization problems (discrete or continuous) have been considered. Zhang et al. [46] addressed inverse shortest paths problems with ℓ_1 norm for p paths using a column generation method. Zhang et al. [47], and Sokkalingam et al. [39] worked on inverse minimum spanning tree problems. Zhang and Liu [48,51] studied inverse linear programming problems. Yang et al. [49] and Zhang and Cai [50] worked on inverse maximum flow and minimum cut problems. Ahuja and Orlin [1,2] also developed a general approach to inverse linear programming problems and discussed applications to some special cases. Ahuja and Orlin [3] proposed combinatorial algorithms for inverse network flow problems. Heuberger [19] gave a survey on inverse combinatorial optimization problems, methods and results.

While most study on inverse optimization so far focuses mainly on combinatorial and network optimization, this type of research has started to expand to nonlinear optimization area. For example, Iyengar and Kang [24] worked on inverse conic programming problems. In this paper, we will study inverse problem of another type of continuous optimization problems: the convex separable programming (CSP) problems. Separable programs are an important class of nonlinear optimization problems which arise from many industrial and managerial areas, for example, production and inventory management [5,8,30,44,52], allocation of resources [5,25,28,53], facility location [42], nonlinear knapsack problems [7,9,31,35,36,38,40,43], agricultural price-endogenous sector modelling [32], stratified sampling [13], optimal design of queueing network models in manufacturing [6], computer systems [16], and so on. Separable programs are nonlinear optimization problems in which the objective function and/or constraints can be expressed as the sum of nonlinear functions, and each nonlinear term involves only one variable. Algorithms for separable programs are widely studied, and interested readers can refer [21,26,27,34,41,45] and references therein.

In this paper, we are interested in the CSP problems in which the objective function has the form $\sum_{i=1}^{n} c_i f_i(x_i)$ where $c_i, i = 1,2,\ldots,n$, may represent costs, capacities or weights, etc.,

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depending on the particular problems. In fact, objective functions in most practical separable programs can be expressed in this form. Some examples of this kind of CSP will be given in Section 2. It will be assumed that $c_i > 0$ for all $i = 1, 2, \ldots, n$. The objective function is convex separable when all the functions $f_i(x_i), i = 1, 2, \ldots, n$, are so.

We shall study in this paper the inverse optimization for CSP with linear constraints. We shall first give sufficient and necessary conditions for a given feasible point of a CSP to be able to become an optimal solution by adjusting its parameter vector $c > 0$ to another one, say $\hat{c} > 0$. When this condition is satisfied, formulations of the inverse optimization are presented in which we find the vector $\hat{c} > 0$ closest to $c > 0$ under the measurement of either ℓ_1 norm or ℓ_2 norm. These inverse optimization problems are either linear programming problems when ℓ_1 norm is used, or quadratic convex separable programming problems when ℓ_2 norm is used. Hence the inverse problems can be solved easily. The rest of the paper is organized as follows. In Section 2, we give some industrial/managerial applications of the inverse CSP, including inverse problems of quality control problem, portfolio optimization problem, and production capacity planning problem. In Section 3, the inverse optimization of CSP with linear equality constraints and nonnegative restrictions on variables is studied. In Section 4, we consider the inverse optimization of CSP with linear inequality constraints and bounds on variables. Conclusions and further research directions are given in Section 5.

2. Some applications of separable programming problems and related inverse optimization

As we mentioned in Section 1, separable programs are widely applied in industrial/managerial areas. In this section we give some examples of separable programs arising in practical fields, and their inverse optimization problems. These include (1) quality control problems in production systems, (2) portfolio selection problems, and (3) production capacity planning problem.

2.1. Quality control in production systems

Consider a manufacturer who wants to control the quality level of production for each time period over a finite planning horizon of n periods, and the quality level of a production facility is defined as the probability of producing a perfect unit. The quality control problem in production systems is to maximize the net profit of the production system with a limitation in the cost of maintaining the desired quality level. This problem can be expressed as the following separable program,

$$\text{QCPS1}\begin{cases} \max & \sum_{i=1}^{n} c_i F_i(x_i) \\ \text{s.t.} & x \in \mathcal{Q} = \left\{ x \,\middle|\, \sum_{i=1}^{n} b_i g_i(x_i) \leqslant D,\ \ell_i \leqslant x_i \leqslant u_i, \right. \\ & \qquad \left. i = 1, 2, \ldots, n \right\} \end{cases} \tag{1}$$

(see [14,37]) where x_i denotes the quality level of the production process during time period $i, c_i > 0$ the expected net profit per unit produced during period $i, F_i(x_i)$ the expected demand function of production at period i, b_i maintenance cost per working hour at period $i, g_i(x_i)$ expected working hours for maintaining the quality level x_i throughout the period i, (hence $b_i g_i(x_i)$ is the estimated cost for maintaining the quality level at x_i in time period i) D an upper limit on the total maintenance cost, and $0 < \ell_i \leqslant x_i \leqslant u_i \leqslant 1$ for all $i = 1, 2, \ldots, n$. It is assumed that for any quality level x_i in period i, the demand function $F_i(x_i)$ is concave and increasing with respect to x_i, and the expected working hours $g_i(x_i)$ is convex and increasing with respect to x_i.

By setting $f_i(x_i) = -F_i(x_i)$, we can rewrite the problem as a minimization problem. In some production systems, manufacturers want the quality levels to be improved at successive time periods. Such requirements can be expressed by the constraints $x_i - x_{i+1} \leqslant 0$. So, we may expand problem (1) to the following form:

$$\text{QCPS2}\begin{cases} \min & \sum_{i=1}^{n} c_i f_i(x_i) \\ \text{s.t.} & x_i - x_{i+1} \leqslant 0, \quad i = 1, 2, \ldots, n, \\ & x \in \mathcal{Q}. \end{cases} \tag{2}$$

However, the quality level is relatively more difficult to adjust than the net profit per unit in some systems, and manufacturers want to make adjustment in the net profit of each period for given quality levels that are determined by the manufacturer according to estimations of market demands. This leads to an important application of inverse optimization to this problem, that is, find $\hat{c} > 0$ which is the closest to the available $c > 0$ and that makes the given quality levels optimal. In fact, the inverse problem of (2) is related to product pricing, since the net profit per unit product in each period is related to the prices of the products and quality levels at different time periods.

Some manufacturers control the production quality level by minimizing the total maintenance cost, meanwhile asking the total net profit to meet a minimum requirement. These problems can be expressed in the following form

$$\text{QCPS3}\begin{cases} \min & \sum_{i=1}^{n} b_i g_i(x_i) \\ \text{s.t.} & x \in \mathcal{Q} = \left\{ x \,\middle|\, \sum_{i=1}^{n} c_i F(x_i) \geqslant C, \right. \\ & \qquad \left. x_i - x_{i+1} \leqslant 0, \ell_i \leqslant x_i \leqslant u_i,\ i = 1, 2, \ldots, n \right\}, \end{cases} \tag{3}$$

where C is the lower bound on the total net profit. The inverse optimization of problem (3) is to find a vector $\hat{b} > 0$, the maintenance cost per working hour for each time period, that is the closest to the estimation $b > 0$, and makes the given quality levels optimal in the sense that they can minimize the total maintenance cost under the restriction to let the quality level meet the target of total profit. This inverse problem is related to pricing per working hour of maintenance for each period i, since maintenance cost has close relations to unit prices and different quality levels at different time periods.

2.2. Portfolio optimization

An investor wants to make decision about an investment on n pre-selected risky assets, for instance, n stocks in a stock market. Let $r \sim \mathcal{N}(\mu, \Sigma)$ denote the random return vector of the n risky assets that are normally distributed with $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ as the expected return vector of the n assets and $\Sigma = (\sigma_{ij}) \in R^{n \times n}$ as the covariance matrix among the asset returns, where the matrix Σ is at least positive semi-definite (usually positive definite), $\sigma_{ij}, i, j = 1, 2, \ldots, n$, are covariance between returns of assets i and j, and $\sigma_{ii} = \sigma_i^2$ the variance of ith asset's return. Suppose the investor has a concave non-decreasing (risk aversion) utility function, and let x_i denote the investment proportion on risky asset i. The portfolio selection problem for one single period is to make a decision on values of the variables $x_i, i = 1, 2, \ldots, n$, such that the expected return from the investment is maximized under a risk level acceptable to the investor. This problem can be expressed as follows: