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Deposit games with reinvestment

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ABSTRACT

In a deposit game coalitions are formed by players combining their capital. The proceeds of their investments then have to be divided among those players. The current model extends earlier work on capital deposits by allowing reinvestment of returns. Two specific subclasses of deposit games are introduced. These subclasses provide insight in two extreme cases. It is seen that each term dependent deposit game possesses a core element. Capital dependent deposit games are also shown to have a core element and even a population monotonic allocation scheme if the revenue function exhibits increasing returns to scale. Furthermore, it is shown that all superadditive games are deposit games if one allows for debt.

1. Introduction

A deposit problem is a decision problem where an individual has a certain amount of capital at his disposal to deposit at a bank. This individual aims to maximise the return on his investment. When we allow for cooperation between individuals by means of joint investment, they can increase their joint return. However, this gives rise to the additional question of how to allocate the proceeds among the individuals. In this paper, we analyse cooperation in deposit situations and in particular, we explore whether an allocation exists such that all players will want to cooperate and form the grand coalition.

Deposit games have previously been studied by Borm et al. (2001), Tan (2000) and De Waegenaere et al. (2005), Borm et al. (2001) define a deposit as a fixed amount of capital at a bank for a certain amount of time, and use a revenue function that describes the revenue of a deposit in a fixed end period, after which no more deposits can be made. In particular, they do not allow for reinvestment of intermediate revenue. Tan (2000) extends this approach and adds the possibility of borrowing.

The approach of De Waegenaere et al. (2005) is more general. They allow reinvestment and a broader range of investment products than deposits. Among other things, the money invested in a single investment product is allowed to change every period, rather than being a fixed amount of capital. The drawback of this approach is that because of its generality, it is harder to draw firm conclusions.

Lemaire (1983) and Izquierdo and Rafels (1996) analyse related issues. Lemaire (1983) gives an overview of the use of game theory in financial issues. Izquierdo and Rafels (1996) analyse games that are related to capital dependent deposit games.

In this paper we take the approach that reinvestment should be possible, while still allowing for a particular form of the revenue function. We start by modelling the decisions of individuals during a discrete and finite number of periods of time. Any deposit will lead to a non-negative revenue. This non-negativity condition is justified because, by the nature of deposits, it is always possible to privately save any amount of money at no cost. There will be no default risk involved in any deposit.

We allow coalitions of individuals to deposit money jointly by cooperating. Clearly, the revenue of a coalition will never be lower than the sum of all individual revenues. We are interested in finding a core element, i.e. we want an allocation for the grand coalition such that no subcoalition has any incentive to split off.

Particular forms of the revenue function are explored by introducing two subclasses: term dependent deposit games and capital dependent deposit games. These classes are also considered by Borm et al. (2001). Each of the classes focusses on one of the two main decisions

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an investor faces when depositing. These main decisions are the amount of capital to deposit and the length of time over which to deposit capital. First we focus on situations where the prime decision is on the length of time; the influence of the amount of capital is assumed to be constant. These situations are called term dependent. We show that the associate term dependent deposit games are totally balanced. The natural counterpart of term dependent deposit games are capital dependent deposit games; the prime decision is on the amount of capital to deposit; depositing over different lengths of time does not add any beneficial options to the investor. We show that, if the rate of return in capital dependent deposit games is increasing in the amount of capital deposited, a core element exists. In fact, we can explicitly construct a population monotonic allocation scheme *a la* Sprumont (1990).

In a further extension of the model, we allow individuals to also have debt, and we show that all non-negative superadditive games are deposit games of this type.

The remainder of this paper is structured as follows. In Section 2 we introduce deposit problems and deposit situations with reinvestment and define corresponding deposit games. In Section 3 we compare the current model to the one analysed by Borm et al. (2001) and show that it indeed is an extension. In the next two sections we analyse two specific subclasses of deposit games. Term dependent deposit games are analysed in Section 4 and in Section 5 capital dependent deposit games are considered. We show how superadditive games can be rewritten as deposit games with reinvestment and debt in Section 6.

2. Deposit problems and deposit games with reinvestment

Similar to Borm et al. (2001), a deposit is defined as a positive, fixed amount of capital c that is at a bank during a prearranged and consecutive number of periods $t_1, t_1+1, \ldots, t_2$, where $1 \leqslant t_1 \leqslant t_2 \leqslant \tau$. Here τ is the final period in which a deposit can be made. The scope of a deposit problem is thus a discrete and finite timespan $\{1, \ldots, \tau\}$. The time interval over which money is deposited is called the term of the deposit

$$T = \{t_1, t_1+1, \ldots, t_2\}.$$

The set of all possible terms of a deposit is given by

$$\mathscr{T} = \{T \subset \{1, \ldots, \tau\} | \exists t_1, t_2 \in \{1, \ldots, \tau\} : T = \{t_1, t_1+1, \ldots, t_2\}\}. \tag{1}$$

A deposit with capital c and term $T = \{t_1, t_1+1, \ldots, t_2\}$ can be represented as a vector over the periods $\{1, 2, \ldots, \tau+1\}$, where at the start of period t_1 an amount c is deposited and at the start of period t_2+1 the amount c is returned. Note that since capital is returned at t_2+1, we extend our model to include the period $\tau+1$.

To illustrate this, we look at a deposit where $\tau = 3$. A possible deposit can be written as $(0, 3, 0, -3)$, which means that three units of capital ($c = 3$) are deposited during periods $t_1 = 2$ and $t_2 = 3$, and this is returned at the beginning of period 4, which is t_2+1.

The set of all possible deposits is denoted

$$\Delta = \{\delta \in \mathbb{R}^{\tau+1} | \exists c > 0, T \in \mathscr{T} : \delta = c \cdot h(T)\}, \tag{2}$$

where we define the function $h : \mathscr{T} \to \mathbb{R}^{\tau+1}$ as a deposit of 1 unit of capital over term $T = \{t_1, t_1+1, \ldots, t_2\}$, so for all $t \in \{1, 2, \ldots, \tau+1\}$ and all $T \in \mathscr{T}$ we have

$$h_t(T) = \begin{cases} 1, & \text{if } t = t_1, \\ -1, & \text{if } t = t_2+1, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

We assume there is a revenue function $P : \Delta \to \mathbb{R}_+^{\tau+1}$ that assigns to each deposit $\delta \in \Delta$ a non-negative revenue in each period of time in $\{1, \ldots, \tau+1\}$. No explicit formula for the revenue function is assumed at this time. Borm et al. (2001) allow revenue to be obtained only in period $\tau+1$. This corresponds to $P_t(\delta) = 0$ for all $t < \tau+1$ and all deposits $\delta \in \Delta$. If we consider again $\tau = 3$ and the deposit $\delta = (0, 3, 0, -3)$ and assume that we get a 4% interest on it, the revenue is $P(\delta) = (0, 0, 0.12, 0.12)$. Note that the revenue from $t = 3$ is not included in the revenue from $t = 4$. In fact, this revenue at $t = 3$ is now available to deposit and will return as such to the individual.

To avoid complications, we assume that money has to be deposited first, and only in a later period a revenue can be attained.

Assumption 2.1. For all $\delta = c \cdot h(\{t_1, t_1+1, \ldots, t_2\}) \in \Delta$ and all $t \leqslant t_1$ we have $P_t(\delta) = 0$.

We also assume that arbitrage is excluded; it is not possible to obtain an infinite revenue using a finite amount of capital.

The capital each individual has available to deposit can come from three sources. The endowments of the individual constitute the first source of capital, which consists for each time period of the income of the individual reduced by the consumption of the individual. The endowments are exogenous and express the net income an individual has available to use for deposits. The second source is payback of capital that was previously deposited. The third source is the revenue from deposits.

The endowments of an individual are given by $m \in \mathbb{R}^\tau$. Although the endowment can be negative at some point in time, we assume that for an individual at each time period the cumulative endowment is non-negative. This boils down to the following assumption.

Assumption 2.2. For all $t \in \{1, \ldots, \tau\}$, we have $\sum_{s=1}^{t} m_s \geqslant 0$.

A *deposit problem* with final period τ, set of deposits Δ, revenue function P and endowments m is denoted by (τ, Δ, P, m).

Because an individual has limited endowments, there are only certain combinations of deposits that he can invest in. A *portfolio* of deposits is denoted by a function $f : \Delta \to \mathbb{N} \cup \{0\}$, which expresses how many units of each deposit are used. A portfolio of deposits an individual is able to finance with his endowments is called feasible. Since the endowment of an individual in a given period is the amount of capital that is available in that period after consumption, and because there is no upper bound on the amount of money that can be deposited, and moreover, the return on any deposit is non-negative, we assume an individual will, in any time period, invest all available capital in deposits.

Note that we take into account the possibility of reinvestment of previously deposited capital and returns. Also note it is possible to get a feasible collection of deposits using the same deposit more than once. It is for instance possible to open two accounts at the bank and deposit the same amount in each account.