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Interfaces with Other Disciplines

Gains from diversification on convex combinations: A majorization and stochastic dominance approach

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ABSTRACT

By incorporating both majorization theory and stochastic dominance theory, this paper presents a general theory and a unifying framework for determining the diversification preferences of risk-averse investors and conditions under which they would unanimously judge a particular asset to be superior. In particular, we develop a theory for comparing the preferences of different convex combinations of assets that characterize a portfolio to give higher expected utility by second-order stochastic dominance. Our findings also provide an additional methodology for determining the second-order stochastic dominance efficient set.

1. Introduction

The pioneer work of Markowitz (1952) and Tobin (1958) on the mean–variance (MV) portfolio selection is a milestone in modern finance theory for optimal portfolio construction, asset allocation, and investment diversification.[1] In the procedure, investors respond to the uncertainty of an investment by selecting a portfolio that maximizes anticipated profit, subject to achieving a specified level of calculated risk or, equivalently, minimizes variance, subject to obtaining a predetermined level of expected gain. However, the disadvantage of using the MV criterion[2] is that it is derived by assuming the von-Neumann and Morgenstern (1944) quadratic utility function and returns being examined are required to be normally distributed or elliptic distributed (Feldstein, 1969; Hanoch and Levy, 1969; Berk, 1997).

To circumvent the limitations of the MV criterion, academics recommend adopting the stochastic dominance (SD) approach, which can be used in constructing a general framework for the analysis of choice and problems of diversification for risk-averse investors under uncertainty without any restriction on the distribution of the assets being analyzed and without imposing the quadratic utility function assumption on investors. Academics have regarded the SD approach as one of the most useful tools for ranking uncertain investment prospects or portfolios because their rankings have been theoretically justified to be equal to the rankings of the corresponding expected utilities. Hanoch and Levy (1969) link stochastic dominance to a class of utility functions for non-satiable and risk-averse investors. Hadar and Russell (1971) develop the analysis using the concept of stochastic dominance and its applicability to choices under conditions of uncertainty, whereas Tesfatsion (1976) further extends their results for diversification using a stochastic dominance approach to maximizing investors' expected utilities. Readers may refer to Ortobelli Lozza (2001) and Post (2008) for an exhaustive overview of other useful results along these lines.

By combining majorization theory with stochastic dominance theory, we extend the theory by developing some new results for choice in portfolio diversification. To specify, we establish some new theorems to determine the preferences of risk-averse investors among different diversified portfolios and show the conditions under which all risk-averse investors would prefer more diversified portfolios to less diversified ones. Our findings are important because they permit investors to specialize the rankings, by second-order stochastic dominance, from among a wide range of convex combinations of assets, and especially because they have implications concerning the weights of allocations. Our findings enable investors to make choices about allocations from their

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[1] To enhance the mean–variance portfolio selection, recently Leung and Wong (2008) apply the technique of the repeated measures design to develop a multivariate Sharpe ratio statistic to test the hypothesis of the equality of multiple Sharpe ratios, whereas Bai et al. (2009, forthcoming) develop new bootstrap-corrected estimations for the optimal return and its asset allocation and prove that these bootstrap-corrected estimates are proportionally consistent with their theoretic counterparts.

[2] This rule provides an excellent approximation to any risk-averse utility function under some restrictions on the range of return; see Levy and Markowitz (1979) for more information.

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capital that result in higher expected utilities. This was one of the topics that Levy (2006) suggested for future research.

In addition, our findings could also be used in determining the second-order stochastic dominance efficient set. Traditionally, there are two decision stages in determining the efficient set; see Bawa et al. (1985). In the first stage, the initial screening of prospects or investments is accomplished by partitioning the feasible set into the efficient and inefficient sets using a stochastic dominance relation.[3] At the second stage, Fishburn's (1974) concept of convex stochastic dominance (CSD) is used to eliminate elements that are not optimal in the sense of CSD. Alternatives that are dominated by convex combinations of other portfolios will be eliminated from the efficient set as they are classified to be inefficient. In this context, our findings allow investors to rank convex combinations of assets by majorization order, which, in turn, implies the rankings of their preferences of second-order stochastic dominance. Thus, our findings assist investors in determining the second-order stochastic dominance efficient set.

Our paper is organized as follows: We begin by introducing definitions and notations and stating some basic properties for the majorization theory and stochastic dominance theory. Section 3 presents our findings on the preferences for risk-averse investors in their choices of diversified portfolios, and Section 4 offers some conclusions.

2. Definitions and notations

In this section, we will first introduce some notations and well-known properties in stochastic dominance theory and majorization theory that we will use in this paper. Considering an economic agent with unitary initial capital, in this paper we study the single-period portfolio selection for risk-averse investors to allocate their wealth to the $n(n>1)$ risks without short selling in order to maximize their expected utilities from the resulting final wealth. Let random variable X be an (excess) return of an asset or prospect. If there are n assets $\vec{X}_n = (X_1, \ldots, X_n)'$, a portfolio of $\vec{X}_n$ without short selling is defined by a convex combination, $\vec{\lambda}_{n'}\vec{X}_n$, of the n assets $\vec{X}_n$ for any $\vec{\lambda}_n \in S_n^0$ where

$$S_n^0 = \left\{ (s_1, s_2, \ldots, s_n)' \in \mathbb{R}^n : 0 \leqslant s_i \leqslant 1 \text{ for any } i, \ \sum_{i=1}^{n} s_i = 1 \right\} \quad (1)$$

in which $\mathbb{R}$ is the set of real numbers. The ith element of $\vec{\lambda}_n$ is the weight of the portfolio allocation on the ith asset of return X_i. A portfolio will be equivalent to return on asset i if $s_i = 1$ and $s_j = 0$ for all $j \neq i$. It is diversified if there exists i such that $0 < s_i < 1$, and is completely diversified if $0 < s_i < 1$ for all $i = 1, 2, \ldots, n$. As we study the properties of majorization in this context, without loss of generality, we further assume that S_n satisfies:

$$S_n = \left\{ (s_1, s_2, \ldots, s_n)' \in \mathbb{R}^n : 1 \geqslant s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n \geqslant 0, \ \sum_{i=1}^{n} s_i = 1 \right\}. \quad (2)$$

We note that the condition of $\sum_{i=1}^{n} s_i = 1$ is not necessary. It could be any positive number in most of the findings in this paper. For convenience, we set $\sum_{i=1}^{n} s_i = 1$ so that the sum of all relative weights is equal to one. In this paper, we will mainly study the properties of majorization by considering $\vec{\lambda}_n \in S_n$ instead of S_n^0.

Suppose that an investor has utility function u, and his/her expected utility for the portfolio $\vec{\lambda}_n'\vec{X}_n$ is $E[u(\vec{\lambda}_n'\vec{X}_n)]$. In this context, we study only the behavior of non-satiable and risk-averse investors whose utility functions belong to the following classes (see, e.g., Ingersoll, 1987):

Definition 1. [4]U_2 is the set of the utility functions, u, defined in $\mathbb{R}$ such that:

$$U_2 = \{u : (-1)^i u^{(i)} \leqslant 0, i = 1, 2\},$$

where $u^{(i)}$ is the ith derivative of the utility function u, and the extended set of utility functions is:

$$U_2^E = \{u : u \text{ is increasing and concave}\}.$$

We note that in the above definition, "increasing" means "non-decreasing". It is known (e.g., see Theorem 11C in Roberts and Varberg, 1973) that u in U_2^E is differentiable almost everywhere and its derivative is continuous almost everywhere. We note that the theory can be easily extended to satisfy utilities defined in Definition 1 to be non-differentiable.[5]

There are many ways to order the elements in S_n. A popular one is to order them by majorization; see, for example, Hardy et al. (1934) and Marshall and Olkin (1979), as stated in the following:

Definition 2. Let $\vec{\alpha}_n, \vec{\beta}_n \in S_n$ in which S_n is defined in (2). $\vec{\beta}_n$ is said to majorize $\vec{\alpha}_n$, denoted by $\vec{\beta}_n \succeq_M \vec{\alpha}_n$, if $\sum_{i=1}^{k} \beta_i \geqslant \sum_{i=1}^{k} \alpha_i$, for all $k = 1, 2, \ldots, n$.

Majorization is a partial order among vectors of real numbers. We illustrate it in the following example:

Example 1. $\left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)' \succeq_M \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)'$ because $\frac{3}{5} > \frac{2}{5}$ and $\frac{3}{5} + \frac{1}{5} \geqslant \frac{2}{5} + \frac{2}{5}$.

Vectors that can be ordered by majorization have some interesting properties. One of them is a Dalton Pigou transfer, as described in the following definition:

Definition 3. [6]For any $\vec{\alpha}_n, \vec{\beta}_n \in S_n$, $\vec{\alpha}_n$ is said to be obtained from $\vec{\beta}_n$ by applying a single Dalton (Pigou) transfer, denoted by $\vec{\beta}_n \xrightarrow{d} \vec{\alpha}_n$, if there exist h and $k (1 \leqslant h < k \leqslant n)$ such that $\alpha_i = \beta_i$ for any $i \neq h, k$; $\alpha_h = \beta_h - \epsilon$; and $\alpha_k = \beta_k + \epsilon$ with $\epsilon > 0$.

For instance, consider the above example that $\vec{\alpha}_3 = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)'$ and $\vec{\beta}_3 = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)'$, As $\alpha_1 = \beta_1 - \frac{1}{5}$, $\alpha_2 = \beta_2 + \frac{1}{5}$, and $\alpha_3 = \beta_3 = \frac{1}{5}$, from Definition 3, we said that $\vec{\alpha}_3$ can be obtained from $\vec{\beta}_3$ by applying a single Dalton transfer by setting $\alpha_1 = \beta_1 - \frac{1}{5}$ and $\alpha_2 = \beta_2 + \frac{1}{5}$. Thus, we write $\vec{\beta}_3 \xrightarrow{d} \vec{\alpha}_3$.

In this example, we also notice that $\vec{\beta}_3$ majorizes $\vec{\alpha}_3$. One may wonder whether there is any relationship between majorization and a Dalton transfer. To answer this question, we have the following theorem:

Theorem 1. *Let $\vec{\alpha}_n, \vec{\beta}_n \in S_n$, $\vec{\beta}_n \succeq_M \vec{\alpha}_n$ if and only if $\vec{\alpha}_n$ can be obtained from $\vec{\beta}_n$ by applying a finite number of Dalton transfers, denoted by $\vec{\beta}_n \xrightarrow{D} \vec{\alpha}_n$.*

Readers may refer to Appendix 1 for the proof of Theorem 1. This theorem states that if $\vec{\beta}_n$ majorizes $\vec{\alpha}_n$, then $\vec{\alpha}_n$ can be obtained from $\vec{\beta}_n$ by applying a finite number of single Dalton transfers, and vice versa. We illustrate the procedure in the following example:

Example 2. Consider $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)'$ and $\left(\frac{4}{5}, \frac{1}{5}, 0\right)'$. As $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)'$ is majorized by $\left(\frac{4}{5}, \frac{1}{5}, 0\right)'$, from Theorem 1, we know that $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)'$ can be obtained by applying a finite number of single Dalton transfers on $\left(\frac{4}{5}, \frac{1}{5}, 0\right)'$. This could be done, for example, by setting $\left(\frac{4}{5}, \frac{1}{5}, 0\right)' \xrightarrow{d} \left(\frac{2}{3}, \frac{1}{3}, 0\right)' \xrightarrow{d} \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)'$. That is, by simply first transferring

[3] Readers may refer to Broll et al. (2006), Wong (2006, 2007) and Wong and Chan (2008) and the references there for more information.

[4] We note that if $u \in U_2$, u is Fréchet differentiable; see, for example, Machina (1982) for more information.

[5] Readers may refer to Wong and Ma (2008) and the references there for more information. In this paper, we will skip the discussion of non-differentiable utilities.

[6] Some academics suggest the reverse direction for the definition of a Dalton Pigou transfer. In this paper, we follow Ok and Kranich (1998) for the definition.