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Continuous Optimization

Enhanced-interval linear programming

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ABSTRACT

An enhanced-interval linear programming (EILP) model and its solution algorithm have been developed that incorporate enhanced-interval uncertainty (e.g., $\mathbf{A}^{\pm}$, $\mathbf{B}^{\pm}$ and $\mathbf{C}^{\pm}$) in a linear optimization framework. As a new extension of linear programming, the EILP model has the following advantages. Its solution space is absolutely feasible compared to that of interval linear programming (ILP), which helps to achieve insight into the expected-value-oriented trade-off between system benefits and risks of constraint violations. The degree of uncertainty of its enhanced-interval objective function (EIOF) would be lower than that of ILP model when the solution space is absolutely feasible, and the EIOF's expected value could be used as a criterion for generating the appropriate alternatives, which help decision-makers obtain non-extreme decisions. Moreover, because it can be decomposed into two submodels, EILP's computational requirement is lower than that of stochastic and fuzzy LP models. The results of a numeric example further indicated the feasibility and effectiveness of EILP model. In addition, EI nonlinear programming models, hybrid stochastic or fuzzy EILP models as well as risk-based trade-off analysis for EI uncertainty within decision process can be further developed to improve its applicability.

1. Introduction

Many of the input parameters in real-world problems exhibit some level of uncertainty due to the scarcity of data (Dantzig, 1995). Traditional linear program (LP) models, however, only address practical problems in which all parameters are deterministic. This includes objective function costs (**C**), constraint coefficients (**A**), and right-hand sides (**B**) (Dantzig, 1955; Chinneck and Ramadan, 2000). Thus, it is important to find ways of using LP methods with intrinsic uncertainties in probabilistic, possibilistic, and/or interval formats (Acevedo and Pistikopoulos, 1998; Liu, 2002; Maqsood et al., 2005). The methods developed to do this could be grouped into stochastic linear programming (SLP), fuzzy linear programming (FLP), interval linear programming (ILP), method, and their hybrid models (Tong, 1994; Huang et al., 1995; Liu, 2002; Chinneck and Ramadan, 2000; Sahinidis, 2004).

For decision-making problems involving randomness, the SLP model deals effectively with various stochastic uncertainties having known or subjective probability distributions (Sengupta et al., 2001; Sahinidis, 2004). The expected value model ($Max\ E[\mathbf{CX}]$, $s.t\ \mathbf{AX} \leqslant \mathbf{B}$, $\mathbf{X} \geqslant 0$), a type of two-stage or multi-stage stochastic programming, can optimize the expected objective functions subject to some expected constraints (Huang and Loucks, 2000; Liu, 2002). Chance-constrained programming ($Max\ \mathbf{CX}$, $s.t\ Pr(\mathbf{AX} \leqslant \mathbf{B}) \geqslant \alpha$, $\mathbf{X} \geqslant 0$) is a second technique for handling this uncertainty; it employs a confidence level for which the stochastic constraint holds (Charnes and Cooper, 1959; Huang, 1996). Dependent-chance programming ($Max\ \Pr(\mathbf{CX} \geqslant \beta)$, $s.t.\ g_j(\mathbf{x}, \xi) \leqslant 0$, $\mathbf{X} \geqslant 0$ for $j = 1, 2, \ldots, p$) is related to maximizing some chance functions of events on stochastic sets in an uncertain and complicated system (Liu, 1997, 2002). However, the increasing data requirements for specifying a discrete or continuous probability density function (PDF) becomes impractical in concrete problems such as water resources or water quality management planning (Huang, 1998). Even in cases when numerous data are available, the computational algorithms such as sampling-based decomposition, approximation schemes, and gradient-based algorithms, may result in complex or nonlinear problems (Birge and Louveaux, 1997). This also restricts the wide application of SLP models (Rockafellar and Roger, 1991; Huang, 1996).

In FLP problems, the objective function and constraints are treated as fuzzy sets with known or subjective membership functions that directly influence the model's solutions (Chang and Wang, 1997; Lodwick and Jamison, 2003). Based on the work of Bellman and Zadeh

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(1970) and Zimmermann (1978), follow-on methods can be classified into two major types: flexible LP and possibilistic LP models (Tanaka and Asai, 1984). The former deals with uncertainties in right-hand sides ($\mathbf{B}$; $\widetilde{Max}\ \mathbf{CX}, s.t.\ \mathbf{AX} \leqslant \widetilde{\mathbf{B}}, \mathbf{X} \geqslant 0$), while the latter reflects uncertainties in $\mathbf{C}$ as well as constraint coefficients ($\mathbf{A}, \mathbf{B}$; $\widetilde{Max}\widetilde{\mathbf{C}}\mathbf{X}, s.t. \widetilde{\mathbf{A}}\mathbf{X} \leqslant \widetilde{\mathbf{B}}, \mathbf{X} \geqslant 0$). In both types of FLP model, the membership function should represent a satisfactory degree of constraint, the decision-maker's expectations about the objective function level, and the range of coefficient's uncertainty (Sahinidis, 2004). However, this has the following shortcomings. It may be difficult to specify membership information for all parameters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ (Sengupta et al., 2001), and most of the FLP solution algorithms depend on control variables (λ) to communicate uncertainty indirectly into the optimization models (Huang et al., 1995; Inuiguchi and Ramík, 2000).

Interval analysis was introduced by Moore in 1959 as a tool for automatic control of the errors in a computed result (Lodwick and Jamison, 2003), where the interval number ($\mathbf{X}^{\pm}$) is considered to be an extension of the real numbers and a subset of the real number line (Moore, 1966). Interval analysis is widely used, and one important application is in solving LP problems with interval coefficients (Ben-Israel and Robers, 1970; Steuer, 1981; Huang and Moore, 1993; Tong, 1994; Chinneck and Ramadan, 2000; Sengupta et al., 2001). This ILP model, as a potential alternative to SLP or FLP, is able to incorporate interval-number uncertainty into the LP model without any assumption of probabilistic or possibilistic distributions (Oliveira and Antunes, 2007).

Ben-Israel and Robers (1970) first introduced a preliminary ILP model for solving a specific LP model whose constraints were the upper and lower bounds ($Max\ \mathbf{CX}\ s.t\ \mathbf{X} \in \{\mathbf{X}\ |\mathbf{B}_1 \leqslant \mathbf{AX} \leqslant \mathbf{B}_2, \mathbf{X} \geqslant 0\}$). Rommelfanger et al. (1989) and Inuiguchi and Sakawa (1995) subsequently proposed a LP method using only the independent upper and lower bounds of an interval objective function (IOF) ($Z^+ = \mathbf{C}^+\mathbf{X}$ and $Z^- = \mathbf{C}^-\mathbf{X}$,). Chanas and Kuchta (1996) and Sengupta et al. (2001) obtained a satisfactory equivalent system for the ILP problem by considering the surrogate objective functions $Max\ Z^{\pm} = \lambda\left(\sum_{j=1}^{n}[c_j^- + \varphi_0(c_j^+ - c_j^-)]x_j\right) + (1-\lambda)\left(\sum_{j=1}^{n}[c_j^- + \varphi_1(c_j^+ - c_j^-)]x_j\right)$ and $Z = 0.5(\mathbf{C}^+ + \mathbf{C}^-)\mathbf{X}$ instead of the original ones. Huang and Moore (1993) and Tong (1994) proposed a new ILP model and BWC method, respectively. BWC converts $Max\ Z^{\pm} = \mathbf{C}^{\pm}\mathbf{X}\ s.t\ \mathbf{X} \in \{\mathbf{X}|\mathbf{A}^{\pm}\mathbf{X} \leqslant \mathbf{B}^{\pm}, \mathbf{X} \geqslant 0\}$ into two submodels: $Z^+ = \mathbf{C}^+\mathbf{X}\ s.t\ \mathbf{X} \in \{\mathbf{X}|\mathbf{A}^-\mathbf{X} \leqslant \mathbf{B}^+, \mathbf{X} \geqslant 0\}$ and $Z^- = \mathbf{C}^-\mathbf{X}\ s.t\ \mathbf{X} \in \{\mathbf{X}|\mathbf{A}^+\mathbf{X} \leqslant \mathbf{B}^-, \mathbf{X} \geqslant 0\}$. Chinneck and Ramadan (2000) extended the BWC method to include nonnegative variables and equality constraints. Although the BWC method does produce the best and worst optimal values, it may result in infeasible decision variable spaces (Huang et al., 1995). Unlike BWC, Huang and Moore's (1993) method is defined generally as $Max\ Z^{\pm} = \mathbf{C}^{\pm}\mathbf{X}^{\pm}\ s.t\ \mathbf{X}^{\pm} \in \{\mathbf{X}^{\pm}|\mathbf{A}^{\pm}\mathbf{X}^{\pm} \leqslant \mathbf{B}^{\pm}, \mathbf{X}^{\pm} \geqslant 0\}$, which provides the solutions of IOF $Z_{opt}^{\pm} = [Z_{opt}^-, Z_{opt}^+]$ and interval decision variables $x_{jopt}^{\pm} = [x_{jopt}^-, x_{jopt}^+]$ for $\forall j$ (Huang et al., 1995; Huang, 1998). It has three major advantages. First, the ILP model incorporates interval information directly into the optimization process. Second, its solution algorithm has lower computational requirements than the SLP and FLP models, and third, the interval solutions produce several alternatives that reflect different decisions (Huang et al., 1995; Chinneck and Ramadan, 2000).

In all these cases, the two extreme decisions (Z_{opt}^- and Z_{opt}^+) simply represent the trade-off between system benefits and risks of $\mathbf{A}^{\pm}$, $\mathbf{B}^{\pm}$ and $\mathbf{C}^{\pm}$'s violations. In the example of municipal waste management in which the objective is to minimize system costs, the decision-maker choosing the lower bound value Z_{opt}^- will end up with a lower system cost but also a higher risk of violating the allowable waste-loading levels (Huang and Moore, 1993). Conversely, the decision to accept a higher system cost by selecting the upper bound value Z_{opt}^+ will correspond to a lower risk. Although decision makers can use Z_{opt}^- and Z_{opt}^+ to help understand extreme alternatives, most wish to focus on the intermediate levels of system costs and risks of violating constraints. Therefore, it is necessary to provide an expected value of the objective function within a relatively narrow interval. To do this, we need a new kind of enhanced-interval (EI) uncertainty that is different than random, fuzzy, and interval variables. The EI variable is defined as random variable in which the upper and lower bounds are known but the PDF is not, although the unknown expected value is considered to lie within the upper and lower bounds. In practice, it is not difficult to determine the EI parameters and variables such as $\mathbf{A}^{\pm}$, $\mathbf{B}^{\pm}$, $\mathbf{C}^{\pm}$ and/or $\mathbf{X}^{\pm}$. For the LP model under EI uncertainty, we could obtain an appropriate interval and the expected value of objective function. Moreover, the feasibility of $x_{jopt}^{\pm} = [x_{jopt}^-, x_{jopt}^+]$ for $\forall j$ has a direct impact on generating different decision alternatives, and thus the constraints in the uncertain LP model should ensure that its solution space is feasible.

As a new extension of interval uncertainty and LP model for generating non-extreme decisions, we define and develop an EI uncertainty and EI linear programming (EILP) model with its solution algorithm. We explain the EILP model and demonstrate its feasibility using a numeric example, and compare EILP with the SLP, FLP, and ILP models.

2. Methodology

2.1. Definition of EI uncertainty

In this subsection, we present several definitions of EI uncertainty and compare the four types of uncertainty.

Definition 1. Let (Ω, Θ, Pr) denote a probability space, where Ω is a nonempty set, Θ is a σ-algebra of subsets of Ω, and Pr is called a probability measure. Random variable y is a measurable function from a probability space (Ω, Θ, Pr) to a bounded, closed subset of real numbers. Then $y^{\pm}$ is defined as an enhanced-interval with known upper and lower bounds and an unknown definite PDF whose expected value $E[\cdot]$ lies between upper and lower bounds:

$$y^{\pm} = [y^-, y^+] = \{s \in \Re | y^- \leqslant s \leqslant y^+\}, \tag{1}$$

$$y^- \leqslant E[y^{\pm}] \leqslant y^+, \tag{2}$$

$$\Phi(x) = Pr\{\omega \in \Omega | y(\omega) \leqslant x\} = \int_{-\infty}^{x} \phi(z)dz, \tag{3}$$

where the PDF $\phi(z) : \Re \to [0, +\infty]$, $\Re$ denotes the set of real numbers, y^+ and y^- are the upper and lower bounds of $y^{\pm}$, respectively, and $\Phi : \Re \to [0, 1]$ is a cumulative probability distribution (CDF). When $y^- = y^+$, $y^{\pm}$ becomes a deterministic number.

Definition 2. Let $\Re^{\pm}$ be a set of EI numbers. An EI vector $\mathbf{Y}^{\pm}$ is a matrix whose elements are EI numbers:

$$\mathbf{Y}^{\pm} = \{y^{\pm} = [y_{ij}^-, y_{ij}^+] | \forall i, j\}, \quad \mathbf{Y}^{\pm} \in \{\Re^{\pm}\}^{m \times n}. \tag{4}$$