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Third-order extensions of Lo's semiparametric bound for European call options

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ABSTRACT

Computing semiparametric bounds for option prices is a widely studied pricing technique. In contrast to parametric pricing techniques, such as Monte-Carlo simulations, semiparametric pricing techniques do not require strong assumptions about the underlying asset price distribution. We extend classical results in this area. Specifically, we derive closed-form semiparametric bounds for the payoff of a European call option, given up to third-order moment (i.e., mean, variance, and skewness) information on the underlying asset price. We analyze how these bounds tighten the corresponding bounds, when only second-order moment (i.e., mean and variance) information is provided. We describe applications of these results in the context of option pricing; as well as in other areas such as inventory management, and actuarial science.

1. Introduction

Going back to the seminal work by Merton (1973), many authors have studied the problem of finding bounds for option prices under incomplete market conditions or an incomplete knowledge of the distribution of the price of the underlying assets (see, e.g., Bertsimas and Popescu (2002), Boyle and Lin (1997), d'Aspremont and El Ghaoui (2006), Grundy (1991), Hobson et al. (2005), Laurence and Wang (2005), Levy (1985), Lo (1987), Perrakis and Ryan (1984), Popescu (2005), Ritchken (1985), and the references therein). Here, we study bounds on the expected payoff of a European call option given only information on the moments of the underlying asset price at maturity. These types of bounds are called *semiparametric bounds*.

Unlike *parametric* pricing techniques, such as Monte-Carlo simulation (cf. Jaeckel, 2002), semiparametric pricing techniques do not require strong assumptions about the underlying asset price distribution. The interest in computing semiparametric bounds for option prices stems mainly from this fact. In particular, semiparametric bounds are used to set bounds for option prices under the risk-neutral measure pricing theory, and to examine the relationship between option prices and the true, as opposed to risk-neutral, distribution of the underlying asset (see, e.g., De la Pena et al., 2004; Boyle and Lin, 1997).

The first, now classical, results in this area were derived by Lo (1987) and Grundy (1991). Lo (1987) gave a closed-form upper bound on the payoff of a European call option when second-order moment (i.e., mean and variance) information about the asset price at maturity is available (see Theorem 2 in Section 2.1). Lo uses the bound as follows: he assumes a certain model for the underlying risk-neutral asset distribution (e.g., lognormal or jump diffusion) and then obtains the risk-neutral moments from this model. The resulting option price bounds, and observed prices are then used to show how sensitive the option prices are to model misspecification. In Section 3.1, we illustrate this approach with a relevant example. Grundy (1991) derived similar upper bounds when only nth order information is given, and gave a simple lower bound on the expected payoff of a European call option when only the mean of the underlying asset price at maturity is known. Grundy uses these semiparametric bounds, as well as Lo's bound, for a quite different purpose. Specifically, instead of assuming knowledge about the underlying risk-neutral asset price moments, he uses observed prices of the option, and semiparametric bounds on the option's payoff, to infer information about the unknown moments (mean and variance) of the risk-neutral distribution of the underlying asset. For this particular type of analysis, it is important to have a simple way to compute the bounds; for example, by having closed-form formulas for them.

The initial work of Lo and Grundy, has been followed by further results in this area, such as the ones developed by Bertsimas and Popescu (2002), Boyle and Lin (1997), d'Aspremont and El Ghaoui (2006), De la Pena et al. (2004), Popescu (2005), and Zuluaga and Peña (2005); to name a few recent ones.

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The goal of this paper is to extend the results of Lo and Grundy. Specifically, we derive closed-form semiparametric bounds for the payoff of a European call option, given up to third-order moment (i.e., mean, variance, and skewness) information on the underlying asset price (see Theorems 5–7 in Section 2). Imposing restrictions on the skewness of the underlying asset distribution is one of the avenues for future research proposed by Grundy (1991, Section VII); and also by Cox et al. (1996, Section 3) in their related work on bounds on expected values of insurance payments. We analyze how these bounds tighten the corresponding bounds, when only up to second-order moment information is provided (Section 2.2). For example, we show that third-order moment information gives tighter bounds on the payoff of a European call option when the option is *close* to being *at the money*. This is precisely the region where the option pricing problem is more interesting. Furthermore, we show that the magnitude of this tightening depends on the relationship between the second and third moments of the underlying asset price. In particular, we prove that if a special relationship between the second and third moment holds, then third-order bounds completely determine the expected payoff of the option (Theorem 5). In his work, Lo (1987, Section 5) concluded "... that option prices obtained from the standard contingent claims analysis may not be robust to misspecification of the fundamental asset's stochastic law of motion"; and proposed the further investigation of this implication as a promising direction for future research. Interestingly, Theorems 5, 6, and Proposition 8 imply that under suitable conditions, option prices are robust to model misspecification, when up to third-order moment information is known. In our exposition, we concentrate on the statements and interpretation of our main results, paying special attention to the semiparametric bounds' values (see Section 2). However, our closed-form approach allow us not only to provide simple algebraic formulas to compute the bounds, but also the distributions (in closed-form) that attain these bounds. These solutions, together with all the proofs; which rely on convex duality (cf. Rockafellar, 1970), are deferred to the Appendix. Applications of our results in the context of option pricing are briefly illustrated in Section 3.1.

Upon the completion of this article, two papers from the insurance literature were brought to our attention; namely, the work of Jansen et al. (1986), and De Schepper and Heijnen (2007). These papers deal with the same, or slight variations, of the problem described above. However, there are some key differences between the results in these papers and the results presented herein. First of all, while the third-order bounds here are given by closed-form formulas (i.e., the bounds have an explicit algebraic dependance on the parameters of the problem), the third-order bounds in Jansen et al. (1986), and De Schepper and Heijnen (2007) are written in terms of the roots of suitable polynomials, which are not explicitly given in terms of the parameters of the problem. Also, we analytically show how the third-order bounds become tighter depending on the relationship between the second and third moment (see Theorem 5, and Proposition 8), and give a detailed characterization of the feasibility conditions of the third-order bound problem (see Proposition 4). Finally, our proof techniques and experiments differ from the ones used in Jansen et al. (1986), and De Schepper and Heijnen (2007). Throughout the article we will make more precise references to the work of Jansen et al. (1986), and De Schepper and Heijnen (2007), as we present our results.

The computation of semiparametric bounds is a classical probability problem (cf. Karlin and Studden, 1966). As a consequence, many related results come from areas other than finance, such as, probability, inventory theory, stochastic programming, supply chain management, and actuarial science. For example, consider the work of Cox (1991), Cox et al. (1996), Bertsimas and Popescu (2005), Bertsimas et al. (2006), Dokov and Morton (2005), Gallego and Moon (1993), He et al. (2007), Natarajan and Linyi (2007), Scarf (1958), Yue et al. (2006), and the references therein. Conversely, our results have applications in these areas, as we show by briefly describing applications in inventory management (see Section 3.2), and actuarial science (see Section 3.3). In particular, we discuss the use of our results to extend the classical result of Scarf (1958) on computing robust inventory levels under incomplete knowledge of the demand distribution. Such extensions are of central importance in the development of the *newsvendor* problem literature (cf. Khouja, 1999, Section 3.5).

It is worth mentioning that all the semiparametric bounds considered here can be numerically computed using semidefinite programming techniques (cf. Todd, 2001). This fact follows from the work of Bertsimas and Popescu (2002), and other related work in the so-called area of *polynomial programming* (see, e.g., Lasserre, 2001, 2002; Zuluaga and Peña, 2005). However, it is well-known that the semidefinite programming model for general semiparametric bounds is highly degenerate and typically suffers from numerical instability problems; especially when the problem is close to being infeasible. Furthermore, semidefinite programming technology is still far from being a standard item in the toolkit of financial academics and practitioners; and formulating the semidefinite program of a semiparametric bound is by no means a straight-forward process (see, e.g., Bertsimas and Popescu, 2002). Our aim here is to obtain closed-form solutions to the semiparametric bound problems being considered. Closed-form solutions are of both practical and theoretical significance, as they allow for easy computation of the bounds, and the performance of sensitivity analysis and optimization over the parameters involved in the problem. Recent examples of the significance of having closed-form formulas for this type of problems are the work of Dokov and Morton (2005), d'Aspremont and El Ghaoui (2006), Laurence and Wang (2005), Hobson et al. (2005), and Yue et al. (2006). For the semiparametric bounds considered here, we obtain closed-form expressions for both the bound's value, and the probability distribution that attains the bound.

An important related area of research is the computation of *arbitrage* bounds, where instead of moments, information about prices of other options on the same underlying asset(s) are used to obtain bounds on the prices of a *target* option. Examples of recent developments in this area are the work of Bertsimas and Popescu (2002), d'Aspremont and El Ghaoui (2006), Hobson et al. (2005), Laurence and Wang (2005), and Vera et al. (2006). The desire to obtain bounds that are tighter than arbitrage bounds, has also lead to recent work on so-called *good deal* bounds (see, e.g., Longarela, 2001; Cochrane and Saa-Requejo, 2000). The computation of semiparametric bounds discussed here, offers another possible avenue for obtaining tighter bounds on option prices.

2. Bounds on the payoff of a European call

In this section, we consider the problem of finding sharp bounds on the expected payoff of a European call option, given information on the first n moments of the underlying asset price at maturity (without making any other assumption on the distribution of the asset price). Finding the sharp upper, and the sharp lower bound for this problem can be (respectively) formulated as the following optimization problems (see, e.g., Bertsimas and Popescu, 2002):

$$(\overline{P}^{\sigma}) \quad \begin{array}{rll} \bar{p}(\sigma) = & \sup & \mathbb{E}_{\pi}((S-1)^{+}) \\ & \text{s.t.} & \mathbb{E}_{\pi}(S^{i}) = \sigma_i \quad \text{for } i = 0, \ldots, n \\ & & \pi \quad \text{is a distribution in } \mathbb{R}_{+}, \end{array}$$