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Interfaces with Other Disciplines

Equilibria of two-sided matching games with common preferences

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ABSTRACT

Problems of matching have long been studied in the operations research literature (assignment problem, secretary problem, stable marriage problem). All of these consider a centralized mechanism whereby a single decision maker chooses a complete matching which optimizes some criterion. This paper analyzes a more realistic scenario in which members of the two groups (buyers–sellers, employers–workers, males–females) randomly meet each other in pairs (interviews, dates) over time and form couples if there is mutual agreement to do so. We assume members of each group have common preferences over members of the other group. Generalizing an earlier model of Alpern and Reyniers [Alpern, S., Reyniers, D.J., 2005. Strategic mating with common preferences. J. Theor. Biol. 237, 337–354], we assume that one group (called males) is r times larger than the other, $r \geqslant 1$. Thus all females, but only $1/r$ of the males, end up matched. Unmatched males have negative utility $-c$. We analyze equilibria of this matching game, depending on the parameters r and c. In a region of (r,c) space with multiple equilibria, we compare these, and analyze their 'efficiency' in several respects. This analysis should prove useful for designers of matching mechanisms who have some control over the sex ratio (e.g. by capping numbers of males at a 'singles event'or by having 'ladies free' nights) or the nonmating cost c (e.g. tax benefits to married couples).

1. Introduction

The problem of pairwise matching of individuals from distinct sets (or sexes) X and Y occurs in many guises: buyers and sellers, employers and employees, medical schools and interns, males and females. We shall use the terminology of the last case, calling the larger group X the males. We assume that individuals of each group have common preferences over whom they would like to be matched with in the other group.

The so-called 'stable marriage' problem proposed by Gale and Shapley (1962) seeks a matching among equal sized finite sets X and Y such that for any two matched pairs (x_1, y_1) and (x_2, y_2), in neither unmatched couple (x_1, y_2) or (x_2, y_1) would each member prefer (with an arbitrary preference relation) their new partner to the one in the original matching. To analyze such questions one must look at complete matchings without considering how they might arise in practice. This 'centralized' problem has received much study (see Roth and Sotomayor, 1990).

More recently, the processes by which complete matchings may arise over time have been analyzed as dynamic games played by the individuals in the two groups. The utilities of these players are often modeled (and will be so here) as 'common preferences' by all members of one sex over individuals of the other. For this reason we can give each individual a 'type' (called x for males and y for females) such that when a couple (x, y) is formed, the male x gets utility y, and the female y gets utility x. We assume that the 'mating season' is short with respect to the time the couple will be together, so that we may ignore the utility consequences of the time (period) in which the couple is formed – there are no search costs in our model. By assuming that an individual's utility is the relative rank of their partner within his or her group, we can normalize these types to the unit interval $[0, 1]$. A male who is unmated at the end of the n'th (final) period gets a utility $-c$, where c is a known parameter representing the cost of failure to mate. In the 'mutual choice', or 'two-sided', models we shall extend in this paper, individuals are randomly paired in each period (that is, the smaller group of females is randomly paired with an equally large randomly chosen set of males – the remaining males are not paired in that period). Then if each member of a matched pair chooses to accept the other rather than go into the next period unmated, they form a couple and are permanently mated. In the final period, players always accept. We call this game $\Gamma_n(r, c)$, where $r \geqslant 1$ (the 'sex ratio') is the initial number of males divided by the initial number of females. This game has been analyzed by Alpern and Reyniers (2005) in the symmetric case $r = 1$. Johnstone (1997) considered a similar dynamic game model and Kalick and Hamilton (1986) simulated a social psychology version. Related games have been studied by Ramsey (in press) and Eriksson et al. (in pressb).

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A strategy for a player in $\Gamma_n(r,c)$ is a rule specifying which potential matches to accept in each period, by determining the least valuable acceptable mate. A strategy profile is called an *equilibrium* if prospective mates are accepted if and only if their type (utility) exceeds the expected utility of the chooser of going into the next period unmated – this is essentially a subgame perfect Nash equilibrium. In the symmetric case $(r=1)$ studied by Alpern and Reyniers (2005), only a single equilibrium was found. In this generalization to $r \geqslant 1$, we find a region of (r,c) space with multiple equilibria. For example, when $n=2$ we find three equilibria: a *choosy* equilibrium, where both groups have high acceptance standards; an *easy* equilibrium, where both groups have low but positive acceptance standards; and a *one-sided* (female choice) equilibrium, where males accept anyone. For $n=2$ (and numerically, for higher n) we find that choosiness at equilibrium goes in the same direction for males and females; equilibria with choosy males have choosy females). We find that the choosy and one-sided equilibria are dynamically stable (attracting fixed points of a dynamical system); but the easy equilibrium is dynamically unstable. We note that the *existence* of an equilibrium follows from a simple application of Brouwer's Fixed Point Theorem in the same way as established for $r=1$ by Alpern and Reyniers (2005). As shown there, equilibria are fully determined by a pair of nonincreasing $n-1$ tuples of threshold values $(u_1, u_2, \ldots, u_{n-1})$ and $(v_1, v_2, \ldots, v_{n-1})$, where u_i is the lowest type female that a top male $(x=1)$ will accept in period i (similarly for v_i for female choice). At equilibrium, a pairing (x,y) in period i will mutually accept and form a couple if and only if $x \geqslant v_i$ and $y \geqslant u_i$. The v_i will always be positive. If all the u_i are 0, we call it a 'one-sided' (or female choice) equilibrium; otherwise we call it a 'two-sided' (or mutual choice) equilibrium.

From the point of view of a single player, a sort of 'secretary problem' (see Ferguson, 1989) is being played out over time, in that he is being presented with a random succession of secretaries. As in the original secretary problem, he may not go back and accept someone he has rejected. However, there are many differences: The distribution in each period depends on previous choices of other players; a secretary may reject him; the objective is expected rank. The closest version of the secretary problem is that of Eriksson et al. (in pressa).

In contrast to two-sided search models such as the well known one of McNamara and Collins (1990), our model is not steady-state. Each period is different: the sex ratio increases and the distribution of types changes according to the strategies employed. The cohorts are initially uniformly distributed but not in any future period. At all equilibria, individuals become less choosy over time, as suggested in the Pennebaker et al. (1979) social science analysis of the country and western song "Don't the girls get prettier at closing time". A good analysis of the effects of changing and uncertain distributions of male quality on female choice has been given in by Collins et al. (2006).

Two-sided matching models have been used in various aspects of economic theory, principally by Burdett and Coles (1997, 1999), Bloch and Ryder (2000), Eeckhout (2000). In biology and psychology, they have been used to describe and analyze mating behavior in animals (Alpern and Reyniers, 1999, Alpern et al., 2005, Bergstrom and Real, 2000), and in humans (Kalick and Hamilton, 1986). Connections with two-sided spatial matching ('rendezvous search') will be discussed in the Conclusions section.

Some notes on terminology. As our model involves two matching processes, the random pairing of unmated individuals at the start of each period and the permanent coupling of pairs who accept each other, we distinguish these by calling the former process *matching* and the latter *mating*. Some results are obtained numerically, and these will be denoted as *Propositions*, covering the region $1 \leqslant r \leqslant 2.5$, $0 \leqslant c \leqslant 2.5$.

The paper is organized as follows. Section 2 gives a complete treatment of the two period problem. We find formulae for the three equilibria: e^1 (one-sided), e^2 (easy), e^3 (choosy). We determine the regions of (r,c) space where they exist (Theorem 1). We show that male and female choosiness vary in the same way at equilibria (Monotonicity Lemma 4). We show that only e^1 and e^3 are dynamically stable (Proposition 5). In Section 3 we use both analytical and numerical methods to establish that these properties of equilibria for $n=2$ periods tend to hold for models with $n>2$ periods.

We wish to thank an anonymous referee of Alpern and Reyniers (2005) for suggesting that an extension of that paper with a non-trivial sex ratio might yield new phenomena – which it has. The addition of the sex ratio has required new techniques to deal with multiple equilibria, as the earlier paper established uniqueness for the trivial (unit) sex ratio case. In addition, the earlier paper dealt only symmetric equilibria, whereas a large part of the story of this paper is about the asymmetries of equilibrium strategies resulting from a skewed sex ratio.

2. The two period game $_2(\boldsymbol{r}, \boldsymbol{c})$

We begin with populations of females and males, with types (quality) uniformly distributed on $[0,1]$. The females have unit density (and unit population), while the males have density (and population) r (the sex ratio) which is at least 1. Let u and v be the male and female first period threshold strategies; females accept a male x iff $x \geqslant v$ while males accept female y iff $y \geqslant u$. A matched male–female pair with types (x,y) will be mated by mutual acceptance if both $x \geqslant u$ and $y \geqslant v$ and with random matching the number (understood as a proportion of the female population) of such couples will be

$$k = (1-u)(1-v), \tag{1}$$

as shown in the unshaded regions of both the female and male populations of Fig. 1. In the left square, females are located according to their type (horizontal y-axis) and the type of the male they are matched with (vertical x-axis). Those in the left rectangle are rejected by their partner and those in the bottom right rectangle reject their partner. The rectangle on the right similarly plots all males, with the additional lower rectangle of unmatched males.

The mean value μ_x of the $r-k$ males x that enter the final period unmated (those not in upper right unshaded rectangle) is calculated by dividing them into those with $x<v$ (of average type $v/2$) and those with $x \geqslant v$ (of average type $(1+v)/2$). The first group of males have population (area) rv, while the second have population $(1-v)$ $(r-1+u)$. Hence

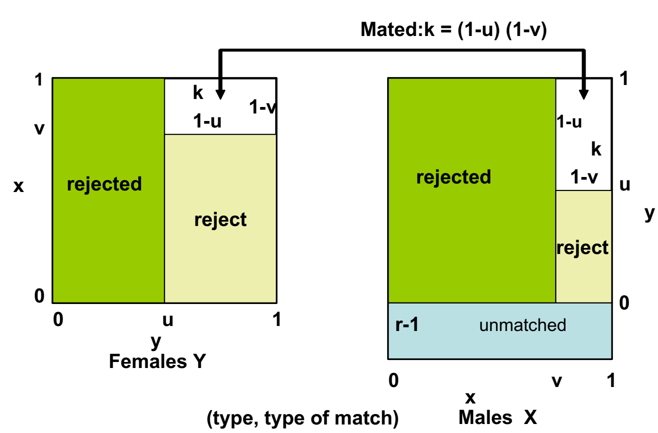

Fig. 1. Couple formation.