European Journal of Operational Research 196 (2009) 1245–1248

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Short Communication

Markowitz's model with Euclidean vector spaces

Salvador Cruz Rambaud [a,*], José García Pérez [b,1], Miguel Ángel Sánchez Granero [c,2], Juan Evangelista Trinidad Segovia [a,3]

[a] Departamento de Dirección y Gestión de Empresas, University of Almería (Spain), La Cañada de San Urbano, s/n, 04071 Almería, Spain
[b] Departamento de Economía Aplicada, University of Almería (Spain), La Cañada de San Urbano, s/n, 04071 Almería, Spain
[c] Departamento de Geometría y Topología, University of Almería (Spain), La Cañada de San Urbano, s/n, 04071 Almería, Spain

ARTICLE INFO

Article history:
Received 9 January 2007
Accepted 11 April 2008
Available online 25 April 2008

Keywords:
Markowitz's model
Portfolio selection
Short sales
Efficient frontier

ABSTRACT

In this paper a new approach of the Markowitz's model is presented. Indeed, using an inner product, a quantitative and explicit solution for optimal portfolio selection is given. To do this, a scalar product is defined in $\Re^n$ which allows us to calculate the composition of the optimal portfolio and the variance for a given expected return by means of the distance between the subspace of feasible solutions and the origin of the affine space.

1. Introduction

The publication in 1952 of the work *Portfolio Selection* by Harry Markowitz in *The Journal of Finance* marked the beginning of the modern portfolio theory. For the first time, the relation between return and risk was included in a financial model together with the concept of rational behaviour of the investors. Several authors, e.g. Lintner (1964), Sharpe (1963, 1970), further developed the portfolio theory using the ideas proposed by Markowitz (1959, 1991), giving rise to the *Diagonal Model* and the *Capital Market Line* (*CML*) (see Bodie and Merton, 2000, for example). In the context of this article it is worth to notice that the original Markowitz model had the "no short sales" constraint. Later this constraint was relaxed (see, for example, Hillier and Lieberman, 2005). A detailed description of the procedure for calculating the optimal portfolio using the classical Markowitz's model can be found in Ødegaard (2006).

Assuming an economy consisting of a set of risky assets together with a single riskless asset, the portfolios along the CML are superior to the efficient frontier portfolios containing risky assets only. There exist several procedures to derive the CML. All of them are based on the Lagrange multipliers method (cf. Merton, 1972, Elton et al., 1976, Elton and Gruber, 1995, Sharpe et al., 1999, Bodie et al., 2000, Ingersoll, 1987, Huang and Litzenberger, 1988, Feldman and Reisman, 2003, Bick, 2004).

Our approach differs from the existing analytical solutions in several aspects:

1. We start with a scalar product defined in the space $\Re^n$ of the portfolio weights for n assets; the so-called inner product is induced by the variance–covariance matrix of the risky assets (a similar approach can be found in Becker (2003)). Such formalization allows obtaining a solution of the Markowitz's model using geometric tools only, by calculating a vector with minimum norm. The main classical results of the Markowitz theory can be derived by simplified calculus comparing to the existing solutions.
2. The scalar product also allows to compute an orthogonal basis, a set of portfolios whose pairwise covariances are zeros, using the Gram-Schmidt algorithm. Therefore, we can always consider a diagonal variance–covariance matrix of security rates which notably simplifies the problem. (A similar idea can be found in Bouchaud and Potters (2004, chapter 12).) This basis has a remarkable importance because the efficient frontier can be derived using a portfolio belonging to the basis and a zero-net investment vector only.
3. We can extend our analysis to the case of risky assets only, as well as to the case when we invest less than 100% of the amount of capital, thus obtaining a superior portfolio.

The organization of this paper is as follows: in Section 2, a scalar product is defined in $\Re^n$ allowing us to calculate the composition of

* Corresponding author. Tel.: +34 950 015 184; fax: +34 950 015 178.
E-mail addresses: scruz@ual.es (S.C. Rambaud), jgarcia@ual.es (J.G. Pérez), misanche@ual.es (M.Á. Sánchez Granero), jetrini@ual.es (J.E. Trinidad Segovia).
[1] Tel.: +34 950 015 173; fax: +34 950 015 472.
[2] Tel.: +34 950 015 311; fax: +34 950 015 481.
[3] Tel.: +34 950 015 817; fax: +34 950 015 178.

0377-2217/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.ejor.2008.04.021

the optimal portfolio for a given level of expected return using the distance between the subspace of feasible solutions and the origin of the affine space. Thus, a formula linking mean and variance is deduced. The main result in this section is that the efficient frontier is a one-dimensional affine subspace of $\Re^n$, where a basis of uncorrelated portfolios can be always constructed. Finally, Section 3 summarizes our findings and concludes.

2. Markowitz's model in a Euclidean vector space

Let us consider a portfolio composed by n assets for which it is possible short sales and purchases without any limit of credit. Thus, the vector space $\Re^n$ can be considered as the set

$$\{(x_1,\ldots,x_n)\in\Re^n : x_i \text{ is the proportion invested in the asset } i\}.$$

In what follows, every vector x (resp. point X) will be represented by means of a $1\times n$ matrix, while its transpose, x^t (resp. X^t), will be a $n\times 1$ matrix.

In the previous space it is defined the following scalar product (the so-called inner product induced by the variance–covariance volatilities matrix of the risky assets):

$$\langle x,y\rangle = x\mathbf{V}y^t,$$

for all $x,y\in\Re^n$, where $\mathbf{V}$ is the matrix of variances–covariances corresponding to the n assets. In effect, taking into account that $\langle x,x\rangle = x\mathbf{V}x^t$ is the variance associated with the composition x of a given portfolio then $\mathbf{V}$ is a symmetric and positive semidefinite matrix. If there exists no composition with null variance, i.e. the common distribution is not concentrated on a hyperplane (practically, this stipulation should be no impediment), it can be deduced that $\langle\cdot,\cdot\rangle$ is a scalar product.

Markowitz's model establishes that, fixing an average return m, the optimal portfolio will be the one with the smallest variance among the portfolios having such average profitability. In other words, once fixed a yield m, the problem is to find a $x\in\Re^n$, such that

- $x\mathbf{V}x^t$, that is to say, the variance, is minimum,
- $\mathbf{R}x^t = m$ or, equivalently, the average return is m, and
- $\mathbf{1}x^t = 1$; in other words, the total investment is 100%,

where $\mathbf{R} = (r_1,\ldots,r_n)$, with r_i the expected return of the asset i ($i = 1,\ldots,n$), and $\mathbf{1} = (1,\ldots,1)$.

In order to do this, let us consider the $(n-1)$-dimensional affine subspaces

$$H_1 = \{Y\in\Re^n : \mathbf{R}Y^t = Y\mathbf{R}^t = m\},$$

with direction linear subspace $U_1 = \{\overrightarrow{AB} : A,B\in H_1\} = \{y : \mathbf{R}y^t = y\mathbf{R}^t = 0\}$, and

$$H_2 = \{Y\in\Re^n : \mathbf{1}Y^t = Y\mathbf{1}^t = 1\},$$

with direction linear subspace $U_2 = \{\overrightarrow{AB} : A,B\in H_2\} = \{y : \mathbf{1}y^t = y\mathbf{1}^t = 0\}$.

The intersection of these hyperplanes is the $(n-2)$-dimensional affine subspace (we assume that $\mathbf{R}$ and $\mathbf{1}$ are linearly independent, i.e. not all assets have the same expected return. Otherwise, the problem is more simple and the reasoning is analogous):

$$S = H_1\cap H_2$$

which will be called the *subspace of feasible solutions*.

So, taking into account that $x = \overrightarrow{OX}$, the problem is reduced to find a point $X\in S$ such that the vector x has a minimum norm $\langle x,x\rangle = x\mathbf{V}x^t$, or equivalently, a point $X\in S$ which represents the minimum distance between 0 and S (see Fig. 1).

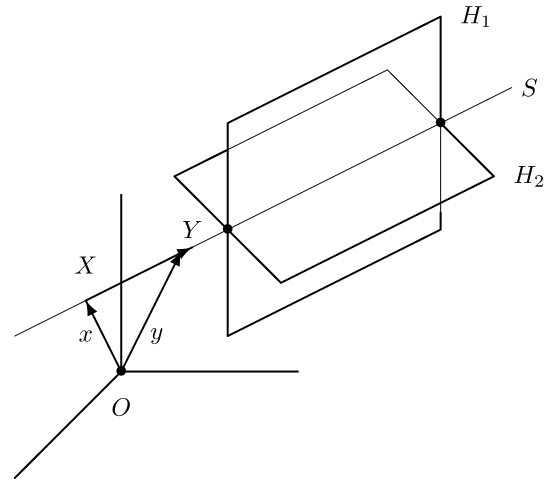

Fig. 1. Minimum distance between O and S.

It is well known that it suffices to find a $X\in S$ such that

$$\langle\overrightarrow{OX},\overrightarrow{XY}\rangle = 0$$

for all $Y\in S$ or, equivalently,

$$\langle x, y-x\rangle = 0.$$

Since $y-x$ ranges the vector subspace underlying S, the problem involves finding a point $X\in S$, such that x is orthogonal to S. Such a point is the intersection of S and the plane spanned by two vectors orthogonal to H_1 and H_2, respectively. A vector orthogonal to H_1 is $\mathbf{R}\mathbf{V}^{-1}$, since

$$\langle z,\mathbf{R}\mathbf{V}^{-1}\rangle = z\mathbf{V}\mathbf{V}^{-1}\mathbf{R}^t = z\mathbf{R}^t = 0$$

for all vector z in the linear subspace U_1, because $\mathbf{R}$ is a vector orthogonal to H_1 according to the Euclidean scalar product. Analogously, a vector orthogonal to H_2 is $\mathbf{1}\mathbf{V}^{-1}$. So, vector x is a linear combination of $\mathbf{R}\mathbf{V}^{-1}$ and $\mathbf{1}\mathbf{V}^{-1}$:

$$x = \alpha_1\mathbf{R}\mathbf{V}^{-1} + \alpha_2\mathbf{1}\mathbf{V}^{-1}, \tag{1}$$

where α_1 and α_2 can be obtained starting from the condition $X\in S$. Hence it is necessary to solve the following system of equations:

$$\begin{cases} x\mathbf{R}^t = \alpha_1\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t + \alpha_2\mathbf{1}\mathbf{V}^{-1}\mathbf{R}^t = m,\\ x\mathbf{1}^t = \alpha_1\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t + \alpha_2\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t = 1,\end{cases}$$

whose solution is

$$\alpha_1 = \frac{m\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t - \mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t}{(\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t)(\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t) - (\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)^2}$$

and

$$\alpha_2 = \frac{\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t - m\mathbf{1}\mathbf{V}^{-1}\mathbf{R}^t}{(\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t)(\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t) - (\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)^2}, \tag{2}$$

and so

$$x = \frac{(m\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t - \mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)\mathbf{R}\mathbf{V}^{-1} + (\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t - m\mathbf{1}\mathbf{V}^{-1}\mathbf{R}^t)\mathbf{1}\mathbf{V}^{-1}}{(\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t)(\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t) - (\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)^2}, \tag{3}$$

which is the composition of the portfolio with average return m and minimum variance. Now, let us calculate an expression for σ^2 according to m. In effect,

$$x\mathbf{V} = \alpha_1\mathbf{R} + \alpha_2\mathbf{1},$$

from which

$$\sigma^2 = x\mathbf{V}x^t = \alpha_1\mathbf{R}x^t + \alpha_2\mathbf{1}x^t$$